Recursive-Cube-of-Rings (RCR) Revisited: Properties and Enhancement

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Abstract: We study recursive-cube-of-rings (RCR), a class of scalable graphs that can potentially provide rich inter-connection network topology for the emerging distributed and parallel computing infrastructure. Through rigorous proof and validating examples, we have corrected previous misunderstandings on the topological properties of these graphs, including node degree, symmetry, diameter and bisection width. To fully harness the potential of structural regularity through RCR construction, new edge connecting rules are proposed. The modified graphs, referred to as *Class-II RCR*, are shown to possess uniform node degrees, better connectivity and better network symmetry, and hence will find better application in parallel computing.
Keywords: recursive-cube-of-rings, inter-connection networks, topology, node degree, diameter, bisection width

I. Introduction

Cloud computing and cloud storage provide unprecedented computing, storage and information processing capabilities. Unlike last century's mainframe, today's cloud computing/storage systems are usually comprised of hundreds of thousands of processing elements (PE) that interconnect and process in a highly parallel, efficient and trust-worthy manner. These parallel computation systems may either operate on shared memory/storage or distributed memory/storage, where the latter scales better and works better for massive processing elements [1].

In a distributed memory/storage system, the PEs connect with and communicate to each other through an *interconnection network* [2]–[4]. Mathematically, an interconnection network is a non-directed graph with no parallel edges or self-loops, where the processing elements (e.g. personal computers) serve as the vertices and the connecting wires (e.g. optical fibers) serves as the edges. To fully harness the power provided by the distributed PEs requires the supporting interconnection network to be judicious organized with efficient communication, low hardware cost, easy applicability of algorithms, strong scalability and fault-tolerance. Although dynamic interconnection (such as switch networks) is also available, most interconnection networks use static interconnection, whose topology is critical to the performance and the cost of the parallel system. For example, a complete graph provides efficient one-hop communication between any two PEs, but planar complete graphs exist for at the most 4 nodes, and a single chip with 5 or more processors must therefore use the more complicated and expensive multi-layer design.

Researchers have developed a number of metrics to measure the goodness of a network topology, reflecting either the performance or the cost or both. For instance, the vertex degree reflects the hardware cost, the bisection width indicates the efficiency of the communication across the network and level of disrupt-tolerance, and the network diameter reveals the maximum communication delay. Additionally, network symmetry in general simplifies the resource management and provides easier means to apply algorithms than an asymmetric topology.

The properties of many basic network models , such as tree, ring, mesh, torus hypercube, butterfly, and de Bruijin networks, have been well studied (e.g. [2], [3], [5]). Using the method of mutation or crossover, these basic models have also been modified, extended, or integrated to provide richer and better interconnection topologies. For example, modified hypercubes are proposed to improve certain topological properties of hypercubes: Folded-hypercube [6] and cross hypercube [7] offer a smaller diameter, meta-cube [8] and ex-

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changed hypercube [9] require a lower hardware cost, and self-similar cubic [10] provides a better scalability, communication and on-chip fabrication possibility. A variety of crossover constructions based on two or more basic network models are also available (e.g. [11]–[13]). Among them is recursive-cube-of-rings (RCR), a family of highly-scalable crossover networks that were originally proposed in [14] and further studied in [15].

An RCR network consists of well-structured cubes and rings, and is generated from recursive expansion of the generation seed. Routing algorithms were subsequently proposed for RCR interconnection networks (e.g. [16]. It was shown in [14], [15] that RCR networks can possess such desirable properties as scalability, symmetry, uniform node degrees, low diameter, and high bisection width. However, caution should be exercised in choosing the parameters for RCR networks, since not all RCR networks enjoy good properties, and many parameters will result in asymmetric and/or unconnected networks.

The inventors defined the RCR network and analyzed some of the most important properties of an interconnection network, including the node degree, the bisection width and the diameter [14]. While the construction of RCR networks through recursive expansion is rather straight-forward, RCR networks can take rather diverse forms depending on the parameters used. The observations and conclusions made in [14] (about node degree, the bisection width and the diameter) spoke for only some RCR cases, and did not cover all the possible scenarios. These flaws were noted by the authors of [15], and improvements were made to the original results. However, a careful investigation reveals that the results provided in [15] about these important properties remain incomplete.

The purpose of this paper is to rectify and improve the results in [14] and [15] and to provide a more complete and accurate characterization of RCR networks. In the first part of the paper, we analyze the node degree (Section III), the bisection width (Section IV) and the diameter (Section V) of the RCR networks, and provide illustrating examples to support our discussion.

In the second part, we further propose a class of modified RCR networks, thereafter referred to as *Class-II RCR* and denoted as RCR-II (Section VI). Through network analysis of connectivity, node degree, network symmetry, diameter, and bisection width, we show that the new class of RCR networks possess better structural and topological regularity than the original RCRs defined in [14]. For example, RCR-II networks have larger bisection widths and shorter network diameters than their RCR counter-parts. An RCR-II network is guaranteed to have uniform vertex degree and, if the parameters are properly chosen, also exhibit desire symmetry property. The findings of this paper will help clarify and correct the misconceptions on the original RCR networks, illuminate a new and better means to exploit the structure of recursive-cube-of-rings, and provide a guideline for choosing good parameters for RCR networks.

II. RCR Networks

A. Introduction of RCR Networks

An RCR network consists of a host of rings connected by cube links [14]. The structure of an RCR network is completely determined by a triple of parameters, the dimension of the cube k, the size of a ring r, and the level of the expansions j from the generation seed, and is thereafter denoted as RCR(k, r, j).

- When $j = 0$, we have the seed network RCR$(k, r, 0)$ from which RCR$(k, r, j \geq 1)$ expands.
- When $r = 1$, the rings degenerates to a single point and the RCR network reduces to a hyper-cube. In other words, RCR networks subsume hyper-cubes as their special case.
- When $k = 0$, the cube vanishes, and the RCR$(0, r, j)$ network becomes a set of $j+1$ disconnected rings each of size r. Such is of little value to parallel computing.

For convenience, in the discussion that follows, we assume $k \geq 1$, $r \geq 1$ and $j \geq 0$.

Let us briefly summarize the structure of an RCR(k, r, j) network [14] and introduce the notations that will be used in the discussion.

An RCR(k, r, j) network has altogether $2^{k+j}r$ nodes in the network. As shown in Figure 1, each node in RCR(k, r, j) is represented by its coordinate, $< a_{k+j-1}, a_{k+j-2}, ..., a_0; b >$, which consists of a cube coordinate $< a_{k+j-1}, a_{k+j-2}, ..., a_0 >$ and a ring coordinate b. An RCR(k, r, j) network is expanded from RCR$(k, r, j-1)$ by replicating RCR$(k, r, j-1)$ twice, preserving all the ring edges, and breaking and reconnecting all the cube edges.

The ring coordinate b, where $0 \leq b \leq r-1$, specifies the position of the node within a ring of dimension r. When $r = 1$ and 2, the ring reduces to a single node and a single line, respectively. When $r > 2$, each node has two distinct ring neighbors that having the same cube coordinate but adjacent ring coordinates,

$$< a_{k+j-1}, a_{k+j-2}, ..., a_0; b \dot{-} 1 >, \quad (1)$$

$$\text{and } < a_{k+j-1}, a_{k+j-2}, ..., a_0; b \dot{+} 1 >, \quad (2)$$

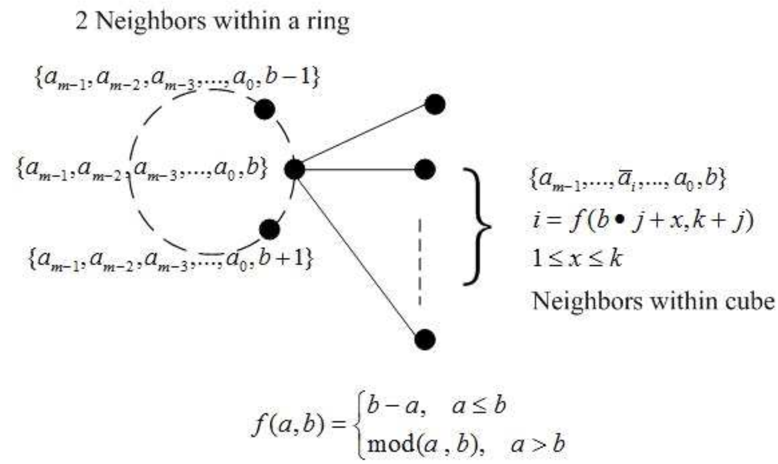


Fig. 1. Construction of an RCR(k, r, j) network

where $\dot{+}$ and $\dot{-}$ stand for the modulo r arithmetic:

$$\alpha \dot{\pm} \beta = \mod (\alpha \pm \beta, r). \tag{3}$$

In general, a node has $\min(r-1, 2)$ ring neighbors.

The cube coordinate $< a_{k+j-1}, a_{k+j-2}, ..., a_0 >$ consists of $k+j$ binary cube bits $a_i \in \{0, 1\}$. Let $\bar{\alpha}$ denote the binary complementary of α, such that $\bar{0} = 1$ and $\bar{1} = 0$. A cube link can only exist between node $< a_{k+j-1}, ..., a_{i+1}, a_i, a_{i-1}, ..., a_0; b >$ and node $< a_{k+j-1}, ..., a_{i+1}, \bar{a_i}, a_{i-1}, ..., a_0; b >$, and it exists only when the index i and the ring coordinate b satisfy the constraint

$$i = f(b \times j + x, k + j), \tag{4}$$

where $0 \leq b \leq r-1$, $1 \leq x \leq k$, and function f is defined as [14]

$$f(a, b) = \begin{cases} b - a, & a \leq b, \\ \mod (a, b), & a > b \end{cases} \tag{5}$$

where a and b are integers. Hence, the number of cube neighbors which a node has is the number of distinct values $f(b \times j + x, k + j)$ may take. It is easy to see that $f(b \times j + x, k + j)$ may take at the most k distinct values.

Having introduced the RCR network as defined in [14], below we discuss the properties of RCR networks.

III. Node Degree

The degree of a node, denoted as D_n, is defined as the total number of distinct ring neighbors and cube neighbors this node has. For an RCR network, $D_n \leq k+2$. It was claimed in [14] and [15] that an RCR(k, r, j) network always had a uniform node degree D_n and a symmetric structure regardless of the parameters used. It was shown that, when the dimension of the rings r was less than or equal to 2, the node degree was $D_n = k + r - 1$ unanimously, and when r was larger than 2, the node degree became $D_n = k+2$ unanimously [14], [15].

However, this conclusion is based on the assumption that each node in an RCR(k, r, j) network has $\min(r-1, 2)$ distinct ring neighbors, and k distinct cube neighbors due to the k possible values of x in (4), where $1 \leq x \leq k$. This assumption is valid for all the RCR examples presented in [14], but does not hold in general. Depending on the choice of the cube dimension k, ring dimension r and the expansion level j, different values of x may generate the same value of $f(b \times j + x, k + j)$ (here the auxiliary variable b is an integer, $0 \leq b \leq r-1$). It is therefore possible for a node to have fewer than k distinct cube neighbors.

Example 1: [Non-uniform node degree of RCR] Consider an RCR$(3, 3, 1)$ network as shown in fig. 2. Following the definition in Subsection II-A and in [14], x may take three possible values: $x \in \{1, 2, 3\}$, and the auxiliary variable b may also take three possible values: $b \in \{0, 1, 2\}$. A node in this RCR network may experience one of the following three scenarios:

- When $b = 0$, the possible values for $f(bj+x, k+j)$ are $3, 2, 1$, which correspond to $x = 1, 2, 3$, respectively. The nodes in this case will have 3 cube neighbors, which lead to a node degree of $D_n = 3 + 2 = 5$.
- When $b = 1$, the possible values for $f(bj + x, k + j)$ are $2, 1, 0$, so the nodes here also have 3 cube neighbors and a node degree of 5.
- When $b = 2$, there are 2 possible values for $f(bj +$

$x, k+j)$: with $x=1$ and $x=3$ we have $f(2\cdot 1+1,4)=1=f(2\cdot 1+3,4)$, and with $x=2$ we have $f(2\cdot 1+2,4)=f(4,4)=0$. The nodes here thus have only 2 cube neighbors and a degree of 4.

Since the node degrees are not uniform, the RCR(3,3,1) network cannot be symmetric.

Theorem 1: [Node degree of RCR] The node degree D_n of an RCR(k,r,j) network satisfies

- When $r \leq 2$, $D_n = k+r-1$ for all the nodes;
- When $r>2$ and at least one of $k \leq j+1$ or $j=0$ is satisfied, $D_n = k+2$ for all the nodes;
- When $r>2$, $j \geq 1$ and $k > j+1$, D_n is not a constant but takes multiple values: $\lceil k/2 \rceil + 2 \leq D_n \leq k+2$.

Proof: Case I: $r \leq 2$. Each node has $r-1$ ring neighbors. Since $bj+x \leq (r-1)j+k \leq j+k$, following the definition of $f(\cdot)$ in (5), we have $f(bj+x,k+j) = (k+j)-(bj+x) = k+(1-b)j-x$. Consider a node with given parameters b,j,k. The function $f(\cdot)$ is a linear function of x and generates k distinct output values for k distinct input values x, indicating that a node always has k cube neighbors. The node degree is therefore $D_n = k+r-1$.

Case II: $r>2$ and $k \leq j+1$. Each node here has 2 ring neighbors. To evaluate the number of cube neighbors, consider separating the nodes in two cases: $b \leq 1$ and $b>1$. (i) When $b \leq 1$, we have $bj+x < j+k$ and hence $f(bj+x,k+j) = k+(1-b)j-x$, which assumes k distinct values for $x=1,2,\cdots,k$. (ii) When $b>1$, we have $bj+x \geq 2j+1 \geq j+k$, and therefore $f(bj+x,k+j) = mod(bj+x,k+j)$, which again generates k distinct values with input $x \in [1,k]$. In either case, a node has k cube neighbors and 2 ring neighbors, making the node degree $D_n = k+2$.

Case III: $r>2$ and $j=0$. Each node has 2 ring neighbors. Since $j=0$, the possible values of $f(bj+x,k) = f(x,k) = k-x$ are $\{k-1,k-2,...,0\}$ for $1 \leq x \leq k$. There exist k distinct values for $f(bj+x,k)$. Therefore, the node degree should be $D_n = k+2$.

Case IV: $r>2$, $j \geq 1$ and $k>j+1$. Each node here has 2 ring neighbors. Separate all the nodes in three cases: $b \leq 1$, $b=2$, or $b>2$. (i) When $b \leq 1$, $bj+x \leq j+x \leq j+k$ and $f(bj+x,k+j) = k+(1-b)j-x$, yielding k distinct values. So each node has k cube neighbors and a degree of $D_n = k+2$. (ii) When $b=2$, since $k-j>1$, we have $bj+k-j = j+k$. There always exists a positive integer $t \leq k-j-1$ and $t \leq j$. Then we have $1 \leq k-j-t < k-j+t \leq k$ and $bj+(k-j-t) < k+j < bj+(k-j+t)$. Let $x_1 = k-j-t$ and $x_2 = k-j+t$. Thus,

$$\begin{aligned} f(bj+x_1) &= f(2j+k-j-t,k+j), \\ &= (k+j)-(j+k-t) = t, \qquad (6) \\ f(bj+x_2,k+j) &= f(2j+k-j+t,k+j), \\ &= mod(k+j+t,k+j) = t. \qquad (7) \end{aligned}$$

Since $x_1 \neq x_2$, the nodes here have fewer than k cube neighbors, and hence their node degree is strictly smaller than $k+2$. Comparing (i) and (ii), we know that the node degree can not be uniform in Case III. (iii) Following the same procedure, we can shown that when $b>2$, the node degree may be either equal to or smaller than $k+2$. In conclusion, when $r>2$, $j \geq 1$ and $k>j+1$, the node degree is not fixed.

From the above discussion, we have known that, for given $b>1$, the node degree may be less than $k+2$. This comes from the fact that some possible values of $bj+x$ are less than $k+j$ and the others are larger than $k+j$. Some $bj+x$ less than $k+j$ will give the same $f(bj+x,k+j)$ with certain $bj+x$ larger than $k+j$. Obviously, the overlapping part is at most $\lfloor k/2 \rfloor$. Therefore, the node degree is always not less than $\lceil k/2 \rceil + 2$.

Remark: It should be noted that a uniform node degree is but a necessary condition for a network to be symmetric. Network symmetry is a stronger condition than merely having a uniform node degree and some apparent regularity in structure. In the case of RCR networks, despite their well-defined structure, a uniform node degree does not necessarily lead to network symmetry.

Example 2: [Uniform-node-degree but asymmetric RCR] The RCR(2,3,2) network shown in Figure 3 has a uniform node degree $D_n = 4$, but is asymmetric. To see this, consider setting an arbitrary node in RCR(2,3,2) as $< 0000; 0 >$ and relabeling all the nodes. From the definition of symmetry, the newly-labeled network will preserve the same connection as the original one. Suppose we relabel node $s =< 0000; 1 >$ as $< 0000; 0 >$. Observe that node s has two neighboring rings: ring B and ring C, and both rings have two cube edges connected with ring A which node s belongs to. It is impossible to find a relabeling scheme for the same network structure that will satisfy the rules (definition) of RCR networks.

IV. Bisection Width

The bisection width, defined as the minimum number of edges that must be removed in order to bisect a

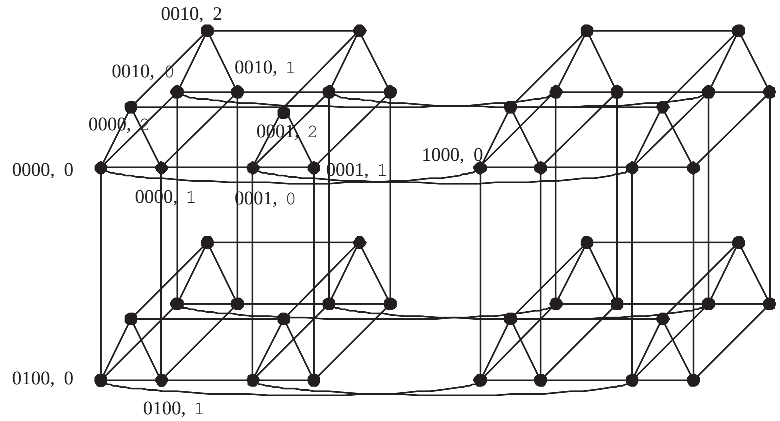


Fig. 2. Construction of RCR(3,3,1) network: nonuniform-node-degree RCR network

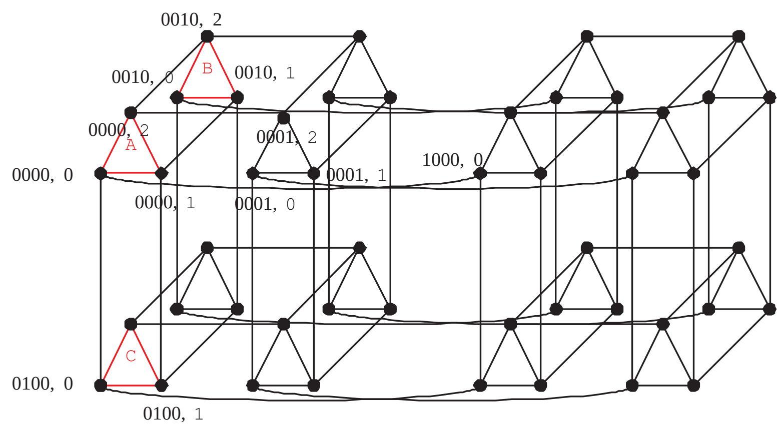


Fig. 3. RCR(2,3,2): A uniform-node-degree but asymmetric network.

network, is another important property for interconnected networks. The conclusion made in [14] on the bisection width of RCR networks was not entirely correct. Some missing cases were picked up and amended in [15], but others remain overlooked.

In [14], the bisection width, $B_{RCR}(k, r, j)$, is computed as

$$B_{RCR}(k, r, j) = Num(k, r, j) \times N/(2 \times r), \quad (8)$$

for all ring dimensions r, where N is the total number of nodes in the networks, and $Num(k, r, j)$ is defined as the number of b values satisfying $f(bj+x, k+j) = k+j-1$.

The authors of [15] recognized that the case of $r = 1$ is an exception. They showed that an RCR$(\cdot, 1, \cdot)$ network comprises two unconnected subnetworks of equal sizes and therefore has 0 bisection width. They thus amended the results in [14] by setting condition $r \geq 2$ on (8), and adding the case of $B_{RCR}(k, 1, j) = 0$. Below we show that more exceptions exist such that an RCR network with $r \geq 2$ may still be unconnected and has 0 bisection width.

Example 3: [An unconnected RCR network with $r = 2$] Consider the RCR(2,2,3) network in Figure 4, whose nodes have coordinates $< a_4, a_3, a_2, a_1, a_0; b >$. When $b = 0$, the possible values of $f(bj+x, k+j)$ are 4 and 3; when $b = 1$, the possible values of $f(bj+x, k+j)$ are 0 and 1. In other words, a node can only have a cube neighbor whose coordinate differs from that itself in one of the four bit positions a_4, a_3, a_1 and a_0. Thus the two sets of nodes, $\{< a_4, a_3, 0, a_1, a_0; b >\}$ and $\{< a_4, a_3, 1, a_1, a_0; b >\}$, each consisting of $2^4 \cdot 2 = 32$ nodes, do not have any inter-connecting edge between them. The network is thus unconnected and has a bisection width of 0.

The reason that [14], [15] failed to spot such cases as Example 3 is that, in computing the bisection width, they always bisected the network into two sub-networks with the $(m-1)$th bit being the complementary of each

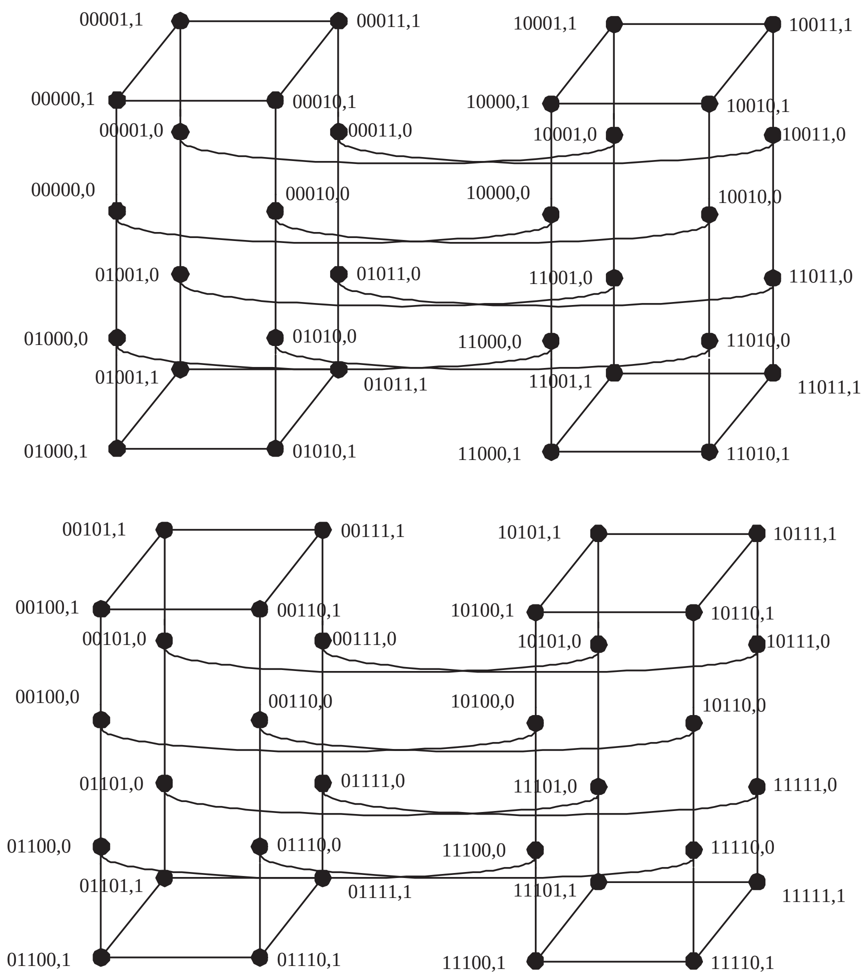


Fig. 4. Construction of RCR(2,3,2) network: unconnected RCR network with $r = 2$

other. However, this cut is not always the minimum cut. To help evaluate the bisection width, let us introduce a new parameter $Num(k, r, j, t)$.

Definition 1: ***[$Num(k, r, j, t)$]*** Consider an RCR(k, r, j) network. For a given integer b if there exists an integer $x \in [1, k]$ such that $f(bj + x, k + j) = t$, we say b satisfies $f(bj + x, k + j) = t$. $Num(k, r, j, t)$ is defined as the number of integer values $b \in [0, k-1]$ that satisfies $f(bj + x, k + j) = t$.

Theorem 2: [Bisection Width of RCR] The bisection width of an RCR(k, r, j) network is upper-bounded by:

$$B_{RCR}(k, r, j) \leq \min_{t \in \{0, \cdots, k+j-1\}} (Num(k, r, j, t)) \times N/(2 \times r), \tag{9}$$

where r is the dimension of rings and N is the total number of nodes.

Proof: Consider bisecting the network nodes into two groups, $\{< a_{k+j-1}, ..., a_{t+1}, a_t = 0, a_{t-1} ..., a_0; b >: a_i \in \{0, 1\} \forall i \text{ except i} \neq \text{t}, \text{b} \in \{0, 1, \cdots, \text{r} - 1\}$ and $\{< a_{k+j-1}, ..., a_{t+1}, \bar{a}_t = 1, a_{t-1}, ..., a_0; b >: a_i \in \{0, 1\} \forall i \text{ except i} \neq \text{t}, \text{b} \in \{0, 1, \cdots, \text{r} - 1\}$, where $0 \leq t \leq k + j - 1$. A cube edge exists between node $< a_{k+j-1}, ..., a_{t+1}, 0, a_{t-1} ..., a_0; b >$ and node $< a_{k+j-1}, ..., a_{t+1}, \bar{a}_t = 1, a_{t-1}, ..., a_0; b >$ if and only if b satisfies $f(b \times j + x, k + j) = t$. Following the definition, there are $Num(k, r, j, t)$ different values of b satisfying $f(b \times j + x, k + j) = t$. Given t and b, there exist $2^{k+j-1} = N/(2r)$ possible values for $a_{k+j-1}, \cdots, a_{t+1}, a_{t-1}, \cdots, a_0$. Therefore, the are altogether $Num(k, r, j, t) \times N/(2r)$ edges between the two groups. Hence, the bisection width is $\min_t (Num(k, r, j, t)) \times N/(2 \times r)$, where $0 \leq t \leq k + j - 1$.

Remark: Theorem 2 considers the case where a bisection cut consists of cube edges only. It is possible for a set of ring edges to also form a bisection cut and to have a smaller size than those formed from cube edges. Hence, what is provided in Theorem 2 represents an upper bound rather than the exact bisection width, as shown in example 4. However, this upper bound is tight, as shown in example 5.

Example 4: [Bisection cut may be formed by ring edges] Consider an RCR$(1, 10, 1)$ network, which comprises 4 rings of dimension 10 each, 10 cube edges connecting node pairs $< c0; b >$ and $< c1; b >$ for $b = 1, 3, ..., 9$ and $c = \{0, 1\}$; and another 10 cube edges connecting node pairs $< 0c; b >$ and $< 1c; b >$ for $b = 0, 2, ..., 8$ and $c = \{0, 1\}$. To bisect the network through cube edges, the minimum cut consists of 10 cube edges. However, the minimum bisect width is 8, resulted from 8 ring edges that connect, say, $< 00; 0 >$ and $< 00; 1 >$, $< 00; 5 >$ and $< 00; 6 >$, $< 01; 5 >$ and $< 01; 6 >$, $< 01; 0 >$ and $< 01; 1 >$, and $< 10; 0 >$ and $< 10; 1 >$, $< 10; 5 >$ and $< 10; 6 >$, $< 11; 5 >$ and $< 11; 6 >$, $< 11; 0 >$ and $< 11; 1 >$, as shown in Fig. 5.

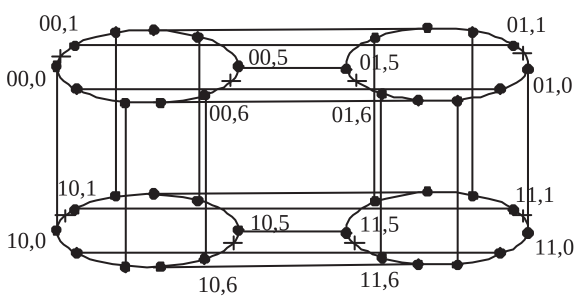


Fig. 5. Construction of RCR(1,10,1) network: Upper bound is not exact bisection width

Example 5: [Upper bound of bisection width is tight] Consider an RCR$(1, 2, 1)$ network as shown in Fig.6. According to Theorem 2, $\min_{t\in\{0,\cdots,k+j-1\}}(Num(k, r, j, t)) = 1$ and $B_{RCR}(k, r, j) \leq 2$. From Fig.6, it is easy to see that the bisection width is exactly 2, which achieves the bound in Theorem 2 with equality.

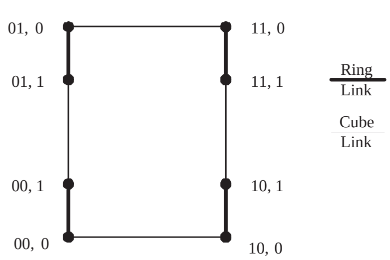


Fig. 6. Construction of RCR(1,2,1) network: upper bound is tight

By definition, an interconnected network is not supposed to be unconnected. The fact that an RCR network may be unconnected (see Example 3) suggests that one needs to exercise with caution in choosing the parameters. Below we present the necessary and sufficient condition that guarantees the connectivity of an RCR network. For convenience, we introduce a new notation $\widehat{\in}$.

Definition: 2 If A can take all the possible integer values between α and β (inclusive), we say that A *covers* the range $[\alpha, \beta]$, and denote it as $A\widehat{\in}[\alpha, \beta]$. Otherwise, we say that range $[\alpha, \beta]$ is not covered by A and denote it as $A\widehat{\notin}[\alpha, \beta]$

Theorem 3: [Sufficient and necessary condition for RCR to be connected] An RCR(k, r, j) network is connected if and only if $f(bj + x, k + j)\widehat{\in}[0, k + j - 1]$ for $0 \leq b \leq r - 1$ and $0 \leq x \leq k$.

Proof: (Sufficient condition: when $f(bj + x, k + j)$ covers all the integer values between 0 and $k + j - 1$, then the RCR network is connected.) Notice that all the nodes having the same cube coordinates (but different ring coordinates) form a ring, and hence we can use the common cube coordinate to identify a ring. To show the connectivity of an RCR(k, r, j) network, it is sufficient to show that any two "adjacent" rings are connected, where by adjacent, we mean that the cube coordinates of the two rings differ in only one bit position. Consider two adj rings with respective cube coordinates $< a_{k+j-1} \cdots a_{t_0+1}0a_{t_0-1} \cdots a_0 >$ and $< a_{k+j-1} \cdots a_{t_0+1}1a_{t_0-1} \cdots a_0 >$. Since $(bj + x, k + j)\widehat{\in}[0, k + j - 1]$, $f(b_0j + x_0, k + j) = t_0$ for some valid values of b_0 and x_0. According to the definition of RCR networks, a cube edge exists that connects node $< a_{k+j-1} \cdots a_{t_0+1}0a_{t_0-1} \cdots a_0; b_0 >$ and node $< a_{k+j-1} \cdots a_{t_0+1}1a_{t_0-1} \cdots a_0; b_0 >$. Hence any source node in the first ring can travel through ring edge(s) to reach node $< a_{k+j-1} \cdots a_{t_0+1}0a_{t_0-1} \cdots a_0; b_0 >$, and then through the cube edge to get to $< a_{k+j-1} \cdots a_{t_0+1}1a_{t_0-1} \cdots a_0; b_0 >$ in the second ring, and again through ring edge(s) to get any destination node in the second ring.

(Necessary condition: when the RCR network is connected, then $f(bj+x, k+j)$ covers all the integer values between 0 and $k+j-1$.) Proof by contradiction. Suppose that the network is connected but there exists $t_0 \in [0, k+j-1]$ such that $f(bj+x, k+j) \neq t_0$ for all the valid values of b and x. According to the definition of RCR networks, there does not exist a cube edge connecting any pair of nodes $< a_{k+j-1} \cdots a_{t_0+1}0a_{t_0-1} \cdots a_0; b >$ and $< a_{k+j-1} \cdots a_{t_0+1}0a_{t_0-1} \cdots a_0; b >$. Hence, an arbitrary node whose coordinate has the former form and an arbitrary node whose coordinate has the latter form are not reachable to each other. For example, node $< 0\cdots 0; 0 >$ cannot reach node $< 1\cdots 1; 0 >$. This contradicts with the connectivity assumption.

Lemma 4: [Necessary condition for RCR to be connected] An RCR(k, r, j) network is unconnected, if $(r-1)k < j$.

Proof: A node in an RCR(k, r, j) network is denoted by the combination of a cube coordinate and a ring coordinate, where the former is a length-$(k+j)$ binary vector, and the latter takes r possible values. According to the cube-edge connecting rule in (5), there are k possible values for x and r possible values for b, and hence at the most kr possible values for $f(bj+x, k+j)$. When $(r-1)k < j$, or, $rk < k+j$, there must be at least one bit index t in the length-$(k+j)$ cube coordinate that does not equal any value of $f(bj+x, k+j)$. Hence, $f(bj+x, k+j) \widehat{\not\supseteq} [0, k+j-1]$. According to theorem 3, the network is therefore unconnected.

Theorem 5: [Parameters for RCR to be connected] An RCR(k, r, j) network is connected, if and only if

$$\begin{cases} (r-1)k \geq j, & r \leq 2 \\ (r-1)k \geq j+1, & r > 2 \end{cases}$$

Proof: From Theorem 3, it is sufficient to show that these parameters, and only these parameters, ensure that function $f(bj+x, k+j) \widehat{\in} [0, k+j-1]$.

Case I: $r = 1$. When $r = 1$ and $(r-1)k \geq j$, then $j = 0$, and the only valid value for $b \in [0, r-1]$ is 0. Thus, $f(bj+x, k+j) = f(x, k) = k - x \widehat{\in} [0, k-1]$ for $x \in [1, k]$. It follows from Lemma 4 that $(r-1)k \geq j$ is the necessary condition for $r = 1$.

Case II: $r = 2$. When $r = 2$ and $(r-1)k \geq j$, then $j \leq k$, and b may take values of either 0 or 1.

$$\begin{aligned} \text{when } b=0, \quad & f(bj+x, k+j) = f(x, k+j), \\ & = k+j-x \widehat{\in} [j, k+j-1], \end{aligned} \tag{10}$$

$$\begin{aligned} \text{when } b=1, \quad & f(bj+x, k+j) = f(x+j, k+j), \\ & = k-x \widehat{\in} [0, k-1]. \end{aligned} \tag{11}$$

Since $j \leq k$, $f(bj+x, k+j) \in [0, k+j-1]$. It follows from Lemma 4 that $(r-1)k \geq j$ is the necessary condition for $r = 2$.

Case III: $r > 2$. We first show that $(r-1)k \geq j+1$ is a sufficient condition for $f(bj+x, k+j) \widehat{\in} [0, k+j-1]$ by differentiating two subcases.

(i) Suppose $k > j$. Then b may take values of $0, 1, \cdots, r-1$.

$$\text{when } b=0,\ f(bj+x, k+j) = (k+j) - x \widehat{\in} [j, j+k-1], \tag{12}$$

$$\begin{aligned} \text{when } b=1,\ & f(bj+x, k+j) = (k+j) - (j+x), \\ & = k - x \widehat{\in} [0, k-1]. \end{aligned} \tag{13}$$

Since $k > j$, we can see $f(bj+x, k+j) \widehat{\in} [0, k+j-1]$.

(ii) Suppose $k \leq j$. From the condition $(r-1)k \geq j+1$, we get $(j+1)/k \leq r-1$. Since j, k, r are integers, we have $\lfloor j/k \rfloor + 1 \leq r-1$. Now b can take values from 0 to $r-1$. We show that as $b = 0, 1, \cdots, \lfloor j/k \rfloor + 1$, the $f(bj+x, k=j) \widehat{\in} [0, k+j-1]$. Since $k \leq j+1$ and $x \in [1, k]$, we have

$$\text{when } b=1, \tag{14}$$
$$f(bj+x, k+j) = (k+j) - (j+x) = k - x \ \widehat{\in} [0, k-1],$$
$$\text{when } b=0, \tag{15}$$
$$f(bj+x, k+j) = (k+j) - x \ \widehat{\in} [j, j+k-1],$$
$$\text{when } b=2, \tag{16}$$
$$f(bj+x, k+j) = mod(2j+x, k+j) \widehat{\in} [j-k+1, j],$$
$$\text{when } b=3, \tag{17}$$
$$f(bj+x, k+j) = mod(3j+x, k+j) \widehat{\in} [j-2k+1, j-k],$$
$$\cdots\cdots$$
$$\text{when } b = \left\lfloor \frac{j}{k} \right\rfloor + 1, \tag{18}$$
$$\begin{aligned} f(bj+x, k+j) &= mod(\left\lfloor \frac{j}{k} \right\rfloor j + j + x, k+j), \\ & \widehat{\in} \left[j - \left\lfloor \frac{j}{k} \right\rfloor k + 1, j - \left\lfloor \frac{j}{k} \right\rfloor k + k \right]. \end{aligned}$$

Since $j - \lfloor \frac{j}{k} \rfloor k + 1 = mod(j, k) + 1 \leq (k-1) + 1 = k$, all the integer segments in the above connect and cover the entire range of $[0, j+k-1]$.

We now show that $(r-1)k \geq j+1$ is also a necessary condition for $f(bj+x, k+j) \widehat{\in} [0, k+j-1]$, by showing that the function f fails to cover $[0, k+j-1]$ otherwise. Again, we evaluate two separate cases:

(i) If $(r-1)k < j$, according to Lemma 4, the RCR(k, r, j) network is unconnected.

(ii) If $(r-1)k = j$,

$$\begin{aligned} \text{when } b=0,\ & f(bj+x, k+j) = f(x, k+j), \\ & = k+j-x \geq j = (r-1)k > k, \end{aligned} \tag{19}$$

$$\begin{aligned} \text{when } b=1,\ & f(bj+x, k+j) = f(j+x, k+j), \\ & = (k+j) - (j+x) = k - x < k, \end{aligned} \tag{20}$$

$$\begin{aligned} \text{when } 2 \leq b \leq r-1,\ & f(bj+x, k+j) = f(bk(r-1)+x, rk), \\ & = mod(bk(r-1)+x, rk) \neq k, \end{aligned} \tag{21}$$

where the first equality in (21) comes from the assumption $(r-1)k = j$, and the second equality comes from the definition of function f. To see that the last inequality in (21) holds, we use proof by contradiction: Suppose there exists $b_0 \in [2, r-1]$ and $x_0 \in [1, k]$ such that $mod(bk(r-1) + x, rk) = k$. That is, we can find an integer B satisfying $bk(r-1) + x = rkB + k$. Since x must be an integer multiple of k in order for the equality to hold, we have $x = k$. The equality now transfers to $b(r-1)+1 = rB+1$, or, $br-b = rB$. Clearly, b must be an integer multiple of r in order for the equality to hold, but $b \in [2, r-1]$, resulting in a conflict. Hence, it follows from (19)-(21) that when $(r-1)k = j$, $f(bj+x, k+j)$ does not produce an output k and hence does not cover $[0, k+j-1]$.

Corollary 6: If the node degree of an RCR(k, r, j) network is non-uniform, then this RCR network is connected.

Proof: From Theorem 1, the node degree of an RCR(k, r, j) network is non-uniform if and only if $r > 2$ and $k > j+1$. This leads to $r > 2$ and $(r-1)k \geq j+1$, and according to Theorem 5, the network is connected.

Remark: Although RCR networks have well-defined and systematic construction, their structure regularity has not been most desirable. From the analysis we performed thus far, (i) An RCR network does not always have uniform node degree. (ii) Even when an RCR network has a uniform degree, it is not necessarily symmetric. (iii) An RCR network having uniform node degree may be unconnected. An RCR network having non-uniform node degree, on the other hand, is always connected.

V. Network Diameter

The diameter of a network measures the minimum number of hops it takes to reach from any node to any other node in the network. [14] stated that the diameter was upper bounded by $k+j-1+\lceil (r-1)/2 \rceil$, which failed to differentiate between connected and unconnected cases. [15] improved the accuracy of the results by recognizing that an RCR network is unconnected and hence has an infinite diameter when the ring dimension r is 1 (assuming the expansion level $j > 0$). When $r > 1$, [15] stated that the diameter was upper bounded by $k + j + 1 + \lfloor r/2 \rfloor$, which, in fact equals $k + j - 1 + \lceil (r-1)/2 \rceil + 2$ for any integer value of r.

However, the results in [15] have not been accurate either. As we have shown in Theorem 5, when $r > 1$, it is also possible for an RCR network to become unconnected and to have an infinite diameter. Further, even in the connected case, the upper-bound provided in [15] is on the optimistic side. The proof in [15] followed the argument that one could always take a one-hop walk from one node to its "cube neighbor" whose cube coordinate differed in one bit position from that of itself and whose ring coordinate was the same as that of itself. Thus, after at the most $k + j + 1$ hops along the cube edges, [15] decided that the source node must have reached an intermediate node that had the same cube coordinate as the destination node. It then took at the most $\lfloor r/2 \rfloor$ hops, or, half the ring dimension, along the ring to reach the destination. This argument is flawed because a pair of nodes having the same ring coordinates and differing in one bit in cube coordinates are not necessarily connected directly. From the definition of RCR networks, a connecting edge exists between two such nodes only when their common ring coordinate b meets the constraint in 5. Here is a counter-example to the conclusion drawn in [15].

Example 6: [Diameter of RCR] Consider an RCR$(2, 5, 7)$ network whose nodes are specified by $< a_8, a_7, \cdots, a_0; b >$ and whose possible values of $f(bj + x, k + j)$ are listed in Table I. Suppose we

	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$
$f(.) =$	$\{8,7\}$	$\{1,0\}$	$\{6,7\}$	$\{4,5\}$	$\{2,3\}$

TABLE I
The possible values of $f(bj + x, k + j)$ for RCR(2,5,7)

want to find the distance between a source node $A =< 00 \cdots 00; 0 >$ to a destination node $B =< 11 \cdots 11; 2 >$. From Table I, we see that each value of ring coordinate b allows the "flip" of only two bits in the cube coordinate. Hence all the possible values of b need to be traversed in order for node A to get to node B. The shortest path is found as follows:

With the ring coordinate $b = 0$, we flip bits a_8 and a_7, i.e. take two hops across the cube edges:

$$< 000000000; 0 > \stackrel{cube-edge}{\rightarrow} < 100000000; 1 > \stackrel{cube-edge}{\rightarrow} < 110000000; 1 > .$$

We next one hop along the ring to change the ring coordinate from $b = 0$ to $b = 4$:

$$< 110000000; 1 > \stackrel{ring-edge}{\rightarrow} < 110000000; 4 >$$

Now with $b = 4$, we flip a_2 and a_3 and then change

b to 3:

$$< 110000000; 4 > \overset{cube-edge}{\to} < 110000100; 4 > \overset{cube-edge}{\to}$$
$$< 110001100; 4 > \overset{ring-edge}{\to} < 110001100; 3 > .$$

Continue hopping alternatively across cube-edges and ring-edges, we get

$$< 110001100; 3 > \overset{cube-edge}{\to} < 110011100; 3 > \overset{cube-edge}{\to}$$
$$< 110111100; 3 > \overset{ring-edge}{\to} < 110111100; 2 > \overset{cube-edge}{\to}$$
$$< 111111100; 2 > \overset{ring-edge}{\to} < 111111100; 1 > \overset{cube-edge}{\to}$$
$$< 111111110; 1 > \overset{cube-edge}{\to} < 111111111; 1 >,$$

at which point, we arrive at the same cube coordinate as node B, indicating that we are now in the same ring as node B. In this example, it then takes one more hope along the ring to get to node B:

$$< 111111111; 1 > \overset{ring-edge}{\to} < 111111111; 2 > .$$

It takes altogether 14 hops to reach from the source node A to the denotation node B, among which 9 are cube-edge hops which change the cube coordinate from $< 000000000 >$ to $< 111111111 >$, 4 are intermediate ring-edge hops which make the change of cube coordinates possible, and 1 is the final ring-edge hop to adjust the ring coordinate. The total number of hops exceeds the upper-bound provided in [15] $k + j + 1 + \lfloor r/2 \rfloor = 2 + 7 + 1 + 2 = 12$.

Theorem 7: [Diameter of RCR]

- If $\min_t(Num(k, r, j, t)) = 0, \quad 0 \le t \le k + j - 1$, then the RCR$(k, r, j)$ network is unconnected with a diameter of ∞.
- If $\min_t(Num(k, r, j, t)) > 0, \quad 0 \le t \le k + j - 1$, then the diameter of RCR(k, r, j) is upper bounded by

$$\begin{cases} k+j+r-1+\lceil \frac{r-1}{2} \rceil = k+j+r-1+\lfloor \frac{r}{2} \rfloor, & r \le 3 \\ k+j+r+\lceil \frac{r-1}{2} \rceil -2 = k+j+r-2+\lfloor \frac{r}{2} \rfloor, & r > 3 \end{cases} \quad (22)$$

This bound is tight in both cases.

Proof: Consider an RCR(k, r, j) network. A pair of nodes have the farthest distance when they stay in two rings whose cube coordinates differ in every bit position. Without loss of generality, suppose the source node has coordinate $< 00 \cdots 00; b = 0 >$ and the destination node has coordinate $< 11 \cdots 11; b = b_0 >$ for some valid value b_0. From Example 6, the worst case involves many intermediate rings, such that one has to go through all the possible values of b in order to find connecting cube edges to reach the destination ring. Depending on where b_0 is closer to $r - 1$ or to 1 in a ring, one may choose to move clockwise or counter-clockwise along the intermediate rings. With at the most $k + j$ cube-edge hops and $r - 1$ ring-edge hops, we will have arrived either at $< 11 \cdots 11; r - 1 >$ or at $< 11 \cdots 11; 1 >$.

Now to move along the destination ring to get to the destination node, we have determine b_0 which has the longest distance with 1 and $r - 1$. If $r \le 3$, then $b_0 = 0$ as shown in 7. If $r > 3$, then $1 < b_0 < r - 1$. As shown in figure 8, the minimum distance between b_0 and $r - 1$ or b_0 and 1 does not exceed $\lfloor \frac{r}{2} \rfloor - 1$. Therefore, it takes no more than a total of $k + j + r + \lfloor \frac{r}{2} \rfloor - 2$ hops to reach from any node to any other node in an RCR(k, r, j) network. It is easy to see that this upper-bound is tight, since Example 6 achieves the bound with equality.

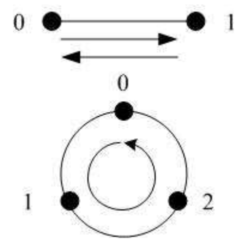


Fig. 7. Long distance between a pair of nodes in RCR for $r \le 3$.

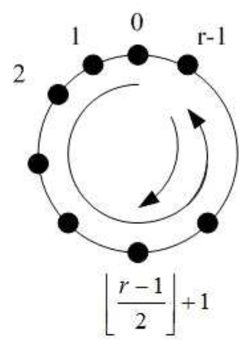


Fig. 8. Long distance between a pair of nodes in RCR for $r > 3$.

VI. Modified RCR Networks

We have thus far revisited RCR networks and rectified the results on node degree, symmetry, connectivity, bisection width and network diameter. A particularly desirable property of RCR networks is their easy construction and high scalability. However, the current edge connecting rules have not fully exploited the potential topological beauty of this class of networks. For example, Theorem 1 states that the node degree of an RCR(k, r, j) network may not be uniform, thus making network symmetry impossible. Further, Lemma 4 and Theorem 5 suggest that high-order RCR networks are doomed to be unconnected, which significantly limits the "useful" scalability of RCR networks. In what follows, we will modify the RCR networks in [14] by redefining the rule for cube edge connection. The modified RCR networks, referred to as Class-II RCR networks, now possess uniform node degree regardless of the parameters used, and hence enjoy a better structural regularity and connectivity.

Definition 3: A Class-II recursive-cube-of-ring, denoted as RCR-II(k, r, j), is determined by three parameters, the cube dimension $k \geq 0$, the ring dimension $r \geq 1$ and the level of expansion $j \geq 0$. The construction of RCR-II(k, r, j) is similar to that of RCR(k, r, j), where node coordinates are represented by:

$$< \underbrace{a_{k+j-1}, a_{k+j-2}, ..., a_0}_{cube\ coordinate}; \underbrace{b}_{ring\ coordinate} > \in \{0,1\}^{k+j} \times \{0, 1, \cdots, r-1\},$$

and the nodes having the same cube coordinates but different ring coordinates belong to the same ring. The only difference between RCR-II and RCR is the connection of cube edges. In RCR-II, node $< a_{k+j-1}, \cdots, a_{t+1}, a_t = 0, a_{t-1}, ..., a_0; b >$ can only be connected to node $< a_{k+j-1}, \cdots, a_{t+1}, a_t = 1, a_{t-1}, ..., a_0; b >$ when the following constraints are satisfied:

$$t = g(bj + x, k + j) \triangleq mod(bj + x, k + j), \quad (23)$$

$$0 \leq b \leq r - 1, \quad 0 \leq x \leq k - 1. \quad (24)$$

Theorem 8: [Node degree of RCR-II] An RCR-II(k, r, j) network has a uniform node degree $D_n = \min(k + r - 1, k + 2)$.

Proof: From linear algebraic, we know that k distinct input values for x will yield k distinct output values for $g(bj + x, k + j) = mod(bj + x, k + j)$. Hence a node in RCR-II always has k cube neighbors. Since every node has $r - 1$ ring neighbors for $r \leq 2$ and 2 ring neighbors for $r > 2$, the result in Theorem 8 thus follows.

Example 7: [Uniform node degree for RCR-II] Example 1 shows that an RCR$(3, 3, 1)$ network has non-uniform node degree. In comparison, RCR-II$(3, 3, 1)$, whose structure is depicted in Fig.9, has a uniform node degree of $D_n = \min(3 + 3 - 1, 3 + 2) = 5$. To see this, note that via the definition of RCR-II networks, x may take three possible values, $x \in \{0, 1, 2\}$, and the auxiliary variable b may also take three possible values: $b \in \{0, 1, 2\}$. A node in this RCR-II network may experience one of the following three scenarios:

- When $b = 0$, the possible values for $f(bj+x, k+j)$ are $0, 1, 2$, which correspond to $x = 0, 1, 2$, respectively. The nodes in this case will have 3 cube neighbors, which lead to a node degree of $D_n = 3 + 2 = 5$.
- When $b = 1$, the possible values for $f(bj+x, k+j)$ are $1, 2, 3$, so the nodes here also have 3 cube neighbors and a node degree of 5.
- When $b = 2$, the possible values for $f(bj + x, k + j)$: are $2, 3, 0$, so the nodes here also have 3 cube neighbors and a node degree of 5

Therefore, the node degree is uniform as shown in Fig. 9. However, this network is not symmetry. Below we discuss conditions for RCR-II to be symmetric.

Theorem 9: [Symmetry of RCR-II] An RCR-II(k, r, j) network is symmetric if $mod(rj, k + j) = 0$.

Proof: To show that RCR-II(k, r, j) is symmetric, we need to show that the network viewed from an arbitrary node $< \alpha_{k+j-1}, ..., \alpha_0; \beta >$ has the same network topology or neighbor-hood connectivity, as viewed from node $< 00 \cdots 00; 0 >$. This is equivalent to finding a proper rule that transforms RCR-II(k, r, j) to itself, such that the $< 00 \cdots 00; 0 >$ is mapped to $< \alpha_{k+j-1}, ..., \alpha_0; \beta >$, and all the other nodes and edges are mapped in a way that preserves the original network topology and cube- and ring-connecting rules.

Let the origin $< 00 \cdots 00; 0 >$ be mapped to the new origin $< \alpha_{k+j-1}, ..., \alpha_0; \beta >$. Assume that an arbitrary node $< a_{k+j-1}, ..., a_i, ..., a_0; b >$ is correspondingly mapped to $< a'_{k+j-1}, ..., a'_i, ..., a'_0; b' >$. We define the transform as follows:

$$\text{ring coordinate} : b' = mod(b + \beta, r); \quad (25)$$

$$\text{cube correlate} : a'_{mod(t-\beta k, k+j)} = a_t \oplus \alpha_{mod(t-\beta k, k+j)}; \quad (26)$$

where the network parameters k, r, j and the new origin $< \alpha_{k+j-1}, \cdots, \alpha_0; \beta >$ are pre-determined constants.

(i) First, this transform is an enclosure, i.e., a valid node coordinate is mapped to a valid node coordinate. From (25), the new ring coordinate b' takes value between 0 and $r - 1$ and is therefore a valid ring coordinate. From (26), the new cube coordinate a'_j has index $0 \leq j \leq k + j - 1$ and takes value $a'_j \in \{0, 1\}$, and is therefore a valid cube coordinate.

(ii) Second, this transform is a one-to-one mapping. From (25) and (26), it is easy to see that if two node have different ring coordinates and/or different cube coordinates before the transform, they will take on different ring coordinates and/or different cube coordinates after transform. Further, since the new ring coordinate is only

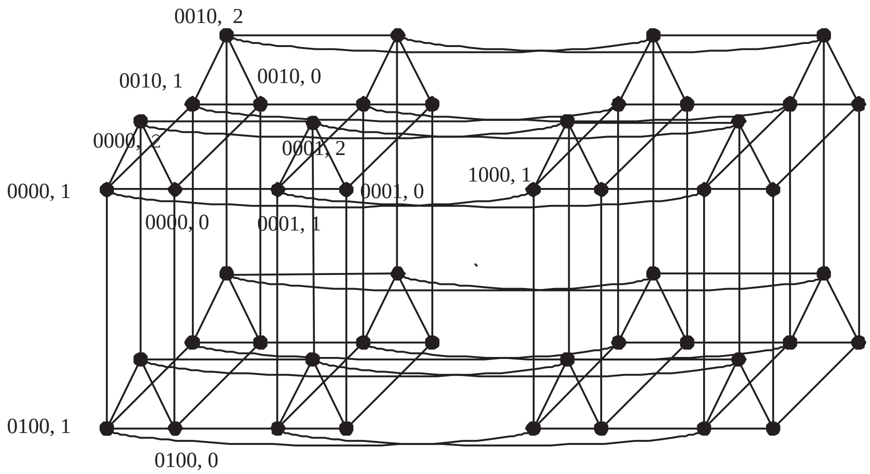


Fig. 9. Construction of RCR-II(3,3,1) network: uniform-node-degree RCR-II network

a function of the old ring coordinate (and predetermined constants) and that the new cube coordinate is only a function of the old cube coordinate (and predetermined constants). If two node have the same ring- or cube-coordinates before the transform, they will take on the same ring- or cube-coordinates after transform.

(iii) Third, the ring edges are preserved under the transform. To see this, consider two nodes in the same ring. These nodes therefore have a common cube coordinate but different ring coordinates. From (ii), after transform, they will take on a common cube coordinate and different ring coordinates, and hence are still in the same ring.

(iv) Finally, the cube edges are preserved under the transform. Suppose there is a cube edge between $< a_{k+j-1}, ..., a_{t+1}, a_t, a_{t-1} ..., a_0; b >$ and $< a_{k+j-1}, ..., a_{t+1}, \bar{a}_t, a_{t-1} ..., a_0; b >$, where, according to the definition of RCR-II networks, $t = mod(bj + x_0, k + j)$ for some integer value of $x_0 \in [0, k - 1]$. Assume $< a_{k+j-1}, ..., a_{t+1}, a_t, a_{t-1} ..., a_0; b >$ is mapped to $< a'_{k+j-1}, ..., a'_{t'+1}, a'_{t'}, a'_{t'-1} ..., a'_0; b' >$ where $t' = mod(t - \beta k, k + j)$. From (26), $< a_{k+j-1}, ..., a_{t+1}, \hat{a}_t, a_{t-1} ..., a_0; b >$ is definitely mapped to $< a'_{k+j-1}, ..., a'_{t'+1}, \hat{a'}_{t'}, a'_{t'-1} ..., a'_0; b' >$. Hence, it is sufficient to show that there exists a cube edge connecting $< a'_{k+j-1}, ..., a'_{t'+1}, a'_{t'}, a'_{t'-1} ..., a'_0; b' >$ and $< a'_{k+j-1}, ..., a'_{t'+1}, \hat{a'}_{t'}, a'_{t'-1} ..., a'_0; b' >$, that is, t' satisfies $t' = mod(b'j + x, k + j)$ for some $x \in [0, k - 1]$ (see the definition of RCR-II networks). We have

$$t' = mod(t - \beta k, k + j), \tag{27}$$
$$= mod((bj + x_0) - \beta k, k + j), \tag{28}$$
$$= mod((bj + \beta j) + x_0 - (\beta k + \beta j), k + j), \tag{29}$$
$$= mod((b + \beta)j + x_0, k + j). \tag{30}$$

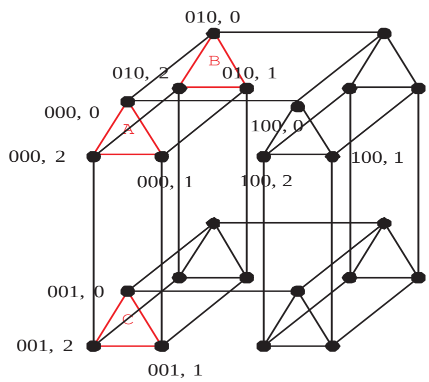
From (25), $b' = b + \beta + Ar$ for some integer A. From the assumption, $mod(rj, k + j) = 0$. Hence,

$$mod((b + \beta)j + x_0, k + j)$$
$$= mod((b + \beta)j + x_0 + Arj, k + j), \tag{31}$$
$$= mod((b + \beta + Ar)j + x_0, k + j), \tag{32}$$
$$= mod(b'j + x_0, k + j). \tag{33}$$

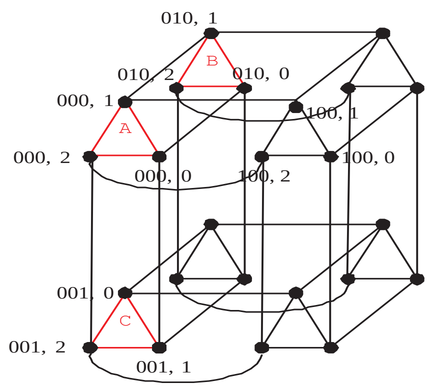
Gathering (30) and (33), we get $t' = mod(b'j + x_0, k + j)$ where $0 \leq x_0 \leq k - 1$. Hence, the cube connectivity is preserved after the transform.

Example 8: [Symmetric RCR-II] To help demonstrate the symmetry (and the balanced structure) of RCR-II networks, compare an RCR and RCR-II network with the same parameters $(2, 3, 1)$ in Fig. 10. It is easy to see that RCR(2,3,1) is asymmetric, since there does not exist a non-distorted mapping that transforms node $< 000, 0 >$ to node $< 000, 1 >$. However, thanks to the different edge connecting rules between ring A, B and C, RCR-II(2,3,1) presents a symmetric network.

Theorem 10: [Sufficient and necessary condition for connectivity of RCR-II] An RCR-II(k, r, j) network is connected if and only if $g(bj + x, k + j) = mod(bj + x, k + j) \widehat{\in} [0, k + j - 1]$ for $0 \leq b \leq r - 1$ and $0 \leq x \leq k - 1$.



(A)asymmetric RCR(2,3,1)



(B)symmetric RCR-II(2,3,1)

Fig. 10. Construction of RCR(2,3,1) and RCR-II(2,3,1).

Proof: The proof follows almost the same procedure as that for Theorem 4, and is therefore omitted.

Theorem 11: [Parameters for RCR-II to be connected] An RCR-II(k,r,j) network is connected, if and only if $(r-1)k \geq j$.

Proof: We show that $(r-1)k \geq j$ is a necessary condition through proof-by-contradiction. Assume the network is connected but $(r-1)k < j$. Since $0 \leq b \leq r-1$ and $0 \leq x \leq k-1$, there are r possible values for b and k possible values for x, and hence at the most rk different values for $t = \mod(bj+x, k+j)$. Since $rk < k+j$, it is impossible for $mod(bj+x, k+j)$ to cover $[0, k+j-1]$, so the network cannot be connected.

We now show that $(r-1)k \geq j$ is also sufficient.

Case I: If $r = 1$ and $(r-1)k \geq j$, then $j = 0$ and $b = 0$. It is easy to see that $g(bj+x, k+j) = mod(x,k)\widehat{\in}[0, k-1]$ for $x \in [0, k-1]$. According to Theorem 10, the network is therefore connected.

Case II: If $r = 2$ and $(r-1)k \geq j$, then $j \leq k$ and $b = 0, 1$. We have $g(bj+x, k+j) = mod(x, k+j)\widehat{\in}[0, k-1]$ when $b = 0$ and $x \in [0, k-1]$, and $g(bj+x, k+j) = mod(j+x, k+j)\widehat{\in}[j, j+k-1]$ when $b = 1$ and $x \in [0, k-1]$. Since $j \leq k$, $g(bj+x, k+j)\widehat{\in}[0, j+k-1]$ and the network is connected.

Case III: If $r > 2$ and $(r-1)k \geq j$, we consider two subcases $k > j+1$ and $k \leq j+1$.

(i) When $k \geq j$ and $x \in [0, k-1]$,

$$\text{when } b = 0, \tag{34}$$
$$g(bj+x, k+j) = mod(x, k+j)\widehat{\in}[0, k-1],$$
$$\text{when } b = 1, \tag{35}$$
$$g(bj+x, k+j) = mod(j+x, k+j)\widehat{\in}[j, k+j-1].$$

Since $j \leq k$, $g(bj+x, k+j)\widehat{\in}[0, k+j-1]$.

(ii) When $k \leq j-1$. Since $(r-1)k \geq j$, $j/k \leq r-1$. Since j, k and r are all integers, $\lceil j/k \rceil \leq r-1$. Since $b \in [0, r-1]$, consider b taking values from $0, 1, \cdots, \lceil j/k \rceil$.

$$\text{when } b = 0,\ g(bj+x, k+j) = mod(x, k+j) \widehat{\in}[0, k-1], \tag{36}$$
$$\text{when } b = 1,\ g(bj+x, k+j) = mod(j+x, k+j) \widehat{\in}[j, j+k-1], \tag{37}$$
$$\text{when } b = 2,\ g(bj+x, k+j) = mod(2j+x, k+j) \widehat{\in}[j-k, j-1], \tag{38}$$
$$\text{when } b = 3,\ g(bj+x, k+j) = mod(3j+x, k+j) \widehat{\in}[j-2k, j-k-1], \tag{39}$$
$$\cdots\cdots$$
$$\text{when } b = \lceil j/k \rceil,\ g(bj+x, k+j) = mod(\lceil j/k \rceil j+x, k+j) \widehat{\in}[j+k-\lceil j/k \rceil k,\ j+2k-\lceil j/k \rceil k-1], \tag{40}$$

Since $j + k - \lceil j/k \rceil k = k(j/k - \lceil j/k \rceil + 1) \leq k$, we can see that $f(bj+x, k+j)\widehat{\in}[0, k+j-1]$. Therefore, the network is connected.

Class-II RCR networks exhibit similar topological properties for the bisection width and the diameter as the original RCR networks.

Theorem 12: [Bisection Width of RCR] The bisection width of an RCR-II(k,r,j) network is upper-bounded by:

$$B_{RCR-II}(k,r,j) \leq \min_{t \in \{0,\cdots,k+j-1\}} Num(k,r,j,t)\frac{N}{2r}, \tag{41}$$

where r is the dimension of rings, $N = r2^{k+j}$ is the total number of nodes, and $Num(k,r,j,t)$ the number of integer values $b \in [0, k-1]$ that satisfies $g(bj+x, k+j) = mod(bj+x, k+j) = t$ for given k, r, j, t, where $x \in [0, k-1]$.

Theorem 13: [Diameter of RCR-II]

- If $\min_t(Num(k,r,j,t))=0,\ 0\le t\le k+j-1$, then the RCR-II$(k,r,j)$ network is unconnected with a diameter of ∞.
- If $\min_t(Num(k,r,j,t))>0,\ 0\le t\le k+j-1$, then the diameter of RCR-II(k,r,j) is upper bounded by

$$\begin{cases} k+j+r-1+\lceil\frac{r-1}{2}\rceil, & r\le 3 \\ k+j+r+\lceil\frac{r-1}{2}\rceil-2, & r>3 \end{cases} \tag{42}$$

This bound is tight in both cases.

Theorems 12 and 13 can be proven using almost the identical arguments as those of RCR networks, and is therefore omitted.

VII. Conclusion

Recursive-cube-of-ring (RCR) networks, proposed by Sun *et al* [14] and further analyzed by Hu *et al* [15], are a rich class of scalable interconnection networks that are determined by three parameters, the ring dimension, the cube dimension and the expansion number. Because of the many available combinations of these three parameters, RCR networks take on a very rich pool of possibilities with rather diverse structures, thus complicating the analysis of their topological properties.

In this paper, we perform a close examination of RCR networks, including the many special cases. Depending on the choice of the parameters, RCR networks may expose rather different properties from each other, some of which are less desirable for parallel computing. For example, for the same seed network (i.e. the same ring diameter and cube diameter), expanding an RCR network an additional level may all of sudden change the network from well-connected to segmented. Our contribution in the first part of this paper is the correction of several misunderstanding and inaccuracies in the previous RCR analysis, including node degree, connectivity, symmetry, diameter and bisection [14], [15]. Validating examples are provided along with the discussion to support our analysis.

Since these RCR networks do not have uniform node degrees nor possess network asymmetry, the second part of the paper focuses on improving and enhancing this class of networks. Our contribution here is the proposition of a class of modified RCR networks, termed RCR-II networks, which preserve the same simplicity, richness and scalability as the original RCRs, but which have uniform node degrees irrespective of the network parameters, and exhibit better connectivity and better symmetry than the original construction. These better properties are achieved with only a simple change of the cube edge connecting rules. Further, since uniform node degree is but necessary condition for a network to be symmetric, sufficient conditions to guarantee a symmetric RCR-II are also derived. Our studies and findings in this paper provide a useful guidance for choosing good parameters for RCR networks.

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