Deffuant Model with General Opinion Distributions: First Impression and Critical Confidence Bound

YILUN SHANG
SUTD-MIT International Design Center, Singapore University of Technology and Design, 20 Dover Drive, 138682, Singapore

Received 13 May 2013; Revised 13 July 2013; accepted 22 July 2013

In the Deffuant model for social influence, pairs of adjacent agents interact at a constant rate and mix up their opinions (represented by continuous variables) only if the distance between opinions is short according to a threshold. We derive a critical threshold for the Deffuant model on $\mathbb{Z}$, above which the opinions converge toward the average value of the initial opinion distribution with probability one, provided the initial distribution has a finite second order moment. We demonstrate our theoretical results by performing extensive numerical simulations on some continuous probability distributions including uniform, Beta, power-law and normal distributions. Noticed is a clear differentiation of convergence rate that unimodal opinions (regardless of being biased or not) achieve consensus much faster than even or polarized opinions. Hereby, the emergence of a single mainstream view is a prominent feature giving rise to fast consensus in public opinion formation and social contagious behavior. Finally, we discuss the Deffuant model on an infinite Cayley tree, through which general network architectures might be factored in. © 2013 Wiley Periodicals, Inc. Complexity 19: 38–49, 2013

Key Words: Deffuant model; social dynamics; consensus; phase transition; Monte Carlo simulation

1. INTRODUCTION

People in everyday life meet and discuss their opinions toward something; they influence one another and as a consequence may adapt their opinions toward other people's opinion. In the last decade, research that focuses on public opinion formation in social networks has gained lots of momentum and varied mathematical opinion dynamics models have been developed. Most established ones include the voter model [1], the majority rule model [2], the social impact model [3], the Sznajd model [4], the Deffuant model [5], and the HK model [6]. We refer the reader to the comprehensive survey [7] for this prolific field.

In this article, we study a continuous opinion dynamics, the Deffuant model [5], where opinion adjustments only proceed when two opinions differ by less in

Correspondence to: Yilun Shang; SUTD-MIT International Design Center, Singapore University of Technology and Design, 20 Dover Drive, Singapore 138682.
E-mail: shylmath@hotmail.com

©2013 Wiley Periodicals, Inc., Vol. 19 No. 2
DOI 10.1002/cplx.21465
Published online 30 September 2013 in Wiley Online Library
(wileyonlinelibrary.com)

magnitude than a given threshold, that is, discussion under bounded confidence [6,8]. Formally, consider a graph $G = (V,E)$ with node set V representing the individuals in a population and edge set E capturing the potential social interaction amongst individuals. Initially, nodes are assigned i.i.d. opinions according to a continuous random variable X. (Typically, $X \sim \mathrm{U}(0,1)$ is uniformly distributed on [0, 1].) Each pair of nodes $\{u,v\} \in E$ meets at the times of a unit rate Poisson process, independent for different pairs. Denote by $X_t(u)$ the opinion value of node $u \in V$ at time t. Thus $X_0(u)$ has the same distribution as X. The above collection of Poisson processes dictates the meeting time of the nodes. When at sometime t the Poisson event occurs at edge $\{u,v\}$ such that the premeeting states of the two nodes are $X_{t-}(u)$ (i.e., $X_{t-}(u) := \lim_{s\uparrow t} X_s(u)$) and $X_{t-}(v)$, we set

$$X_t(u)=\begin{cases} X_{t-}(u)+\mu(X_{t-}(v)-X_{t-}(u)) & \text{if } |X_{t-}(u)-X_{t-}(v)| \le d; \\ X_{t-}(u) & \text{otherwise,} \end{cases} \tag{1}$$

and

$$X_t(v)=\begin{cases} X_{t-}(v)+\mu(X_{t-}(u)-X_{t-}(v)) & \text{if } |X_{t-}(u)-X_{t-}(v)| \le d; \\ X_{t-}(v) & \text{otherwise,} \end{cases} \tag{2}$$

where $\mu \in (0, 1/2]$ is referred to as the convergence parameter and $d \in \mathbb{R}$ is the so-called confidence bound [6]. The updating rule (1) and (2) roughly means that the opinions of the interacting individuals shift toward each other by a relative amount μ when they meet and find their opinion difference is small enough. A value of $\mu=1/2$ means that the two discussing individuals will meet halfway.

Results on the Deffuant dynamics are mostly presented under the assumption that $X \sim \mathrm{U}(0,1)$, namely the initial opinions are uniformly and randomly chosen in the range [0, 1]. In this scenario, it is shown through Monte Carlo simulations that [9,10], for $d > d_c=1/2$, all individuals eventually share the same opinion $1/2$ on a variety of networks, be them complete graphs, regular lattices, random graphs or even scale-free networks. When d becomes smaller than $1/2$, the complete consensus can not be achieved and numerical simulations again unravel that [5,11] the number of clusters in the final configuration can be approximated by $1/(2d)$. The parameter μ only influences the speed of convergence [5].

One issue of interest concerns the unequal initial opinion distribution: what would happen if the initial opinions were unevenly distributed or had preference/bias? Laslier [12] conjectured that the opinions will tend to average initial opinion. Through extensive simulations on a directed Barabási-Albert network, Jacobmeier [13] found that the final consensus value is always around the mean of the initial opinions (which is dubbed as "first impression"), leading to the conclusion that for a communicative social community "the first impression guides the opinion forming" [13]. Nonuniformly distributed opinions are simulated recently in [14]. The authors argue that initial opinion plays a key role in the collective opinion evolution and the final opinions converge more easily when the initial ones are closer.

This sort of first impression effect is commonplace in the real life. In this article, we aim to provide rigorous analytical results to support this first impression observation. More specifically, we consider the Deffuant model on the real line $\mathbb{Z}$. As long as the initial distribution has finite second moment, namely $\mathrm{E}(X^2) < \infty$, we show that there exists a critical confidence bound d_c such that, when $d > d_c$,

$$\mathrm{P}\Big(\lim_{t\to\infty} X_t(u)=\mathrm{E}\,X_0(u)=\mathrm{E}\,X\Big)=1 \tag{3}$$

for all $u \in \mathbb{Z}$. Although, there has been a plethora of numerical results on the Deffuant model (and its variants) in sociophysics, the mathematical analyses are done only very recently by Lanchier [15] and Häggström [16] independently using different methods. They considered the one-dimensional Deffuant model on $\mathbb{Z}$ with uniform initial opinion distribution in the interval [0, 1]. A consequent critical confidence bound $d_c = 1/2$ was obtained. Here, we follow the line of Häggström's work by first establishing a pairwise average procedure (called sharing a drink (SAD) [16]) on $\mathbb{Z}$ and then deriving the critical value d_c with the help of SAD procedure and ε-flat points concept. Our main result is summarized in Theorem 1 (see section 3 below).

Taking some concrete continuous distributions such as uniform, Beta, power-law, and normal distributions as the initial opinion distribution, we provide the exact expressions of critical confidence bound d_c for them (see Table 1 below). In section 4, extensive simulations are performed to illustrate the availability of our theoretical results. Interestingly, we observe that either even or polarized opinions in the initial configuration may delay the process of forming collective opinion, which agrees with our lay intuition that widely divergent topics and controversial issues are difficult (take much longer time!) to seek consensus on. In the meantime, unimodal opinions (either unbiased or biased) are prone to achieve consensus fast, indicating that the emergence of a single mainstream view is critical to guiding consensus formation efficiently.

Finally, in the Discussion section, we consider the applicability and limitations of our methodology by treating the Deffuant model on an infinite Cayley tree, which is hoped to bridge the gap between path and more complex and realistic networks.

DOI 10.1002/cplx

TABLE 1

Properties for Four Types of Opinion Distributions: $U(a,b)$ is the Uniform Distribution on the Interval [*a, b*], $\mathrm{Beta}(\alpha,\beta)$ is the Beta Distribution on the Interval [0, 1] with $\alpha,\beta>0$, $\mathrm{PL}(x_0,\gamma)$ is the Power-Law (Pareto) Distribution on $[x_0,\infty)$ with $x_0,\gamma>0$, and $N(\hat{\mu},\hat{\sigma}^2)$ is the Normal Distribution on $\mathbb{R}$

Opinion Distribution	First Impression	Second Order Moment	Critical Confidence Bound
X	$\mathrm{E}\,X$	$\mathrm{E}\,(X^2)$	d_c
$U(a,b)$	$\frac{a+b}{2}$	$\frac{a^2+ab+b^2}{3}$	$\frac{b-a}{2}$
$\mathrm{Beta}(\alpha,\beta)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha^2(\alpha+1)+\alpha\beta(\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$\frac{\max\{\alpha,\beta\}}{\alpha+\beta}$
$\mathrm{PL}(x_0,\gamma)$	$\frac{x_0\gamma}{\gamma-1}$	$\frac{x_0^2\gamma}{\gamma-2}$ (when $\gamma>2$)	∞
$N(\hat{\mu},\hat{\sigma}^2)$	$\hat{\mu}$	$\hat{\mu}^2+\hat{\sigma}^2$	∞

They all have finite second order moments satisfying the assumption of Theorem 1.

2. PRELIMINARIES

2.1. SAD Procedure

In this section, we recall the methodology of Häggström [16], which is applicable to our general situation. First define

$$Y_0(u)=\begin{cases}1 & \text{for } u=0;\\ 0 & \text{for } u\in\mathbb{Z}\backslash\{0\}.\end{cases} \tag{4}$$

A discrete-time process $\{Y_i(u)\}_{u\in\mathbb{Z}}$ for $i\geq 0$ can be defined iteratively as follows. Given a sequence $u_1,u_2,\cdots\in\mathbb{Z}$ and $\mu\in(0,1/2]$, we obtain the configuration $\{Y_i(u)\}_{u\in\mathbb{Z}}$ by letting

$$Y_i(u)=\begin{cases}Y_{i-1}(u)+\mu(Y_{i-1}(u+1)-Y_{i-1}(u)) & \text{for } u=u_i;\\ Y_{i-1}(u)+\mu(Y_{i-1}(u-1)-Y_{i-1}(u)) & \text{for } u=u_i+1;\\ Y_{i-1}(u) & \text{for } u\in\mathbb{Z}\backslash\{u_i,u_i+1\}.\end{cases} \tag{5}$$

This process is called SAD, which can be viewed as a liquid-exchanging procedure for glasses located at each site $u\in\mathbb{Z}$. Initially, only the glass at 0 is full, namely $Y_0(0)=1$, while all others are empty, namely, $Y_0(u)=0$ for $u\neq 0$. At each subsequent time step i, we pick two adjacent glasses at u_i and u_i+1, and pouring liquids from the glass with higher level to that with lower level by a relative amount μ. The following are a couple of basic properties regarding SAD [16].

Lemma 1. (*monotonicity*)
Suppose that $\{Y_i(u)\}_{u\in\mathbb{Z}}$ is obtained via a SAD such that $u_j\neq -1$ for all $1\leq j\leq i$. Then $Y_i(0)\geq Y_i(1)\geq Y_i(2)\geq\cdots$.

Lemma 2. (*unimodality*)
Suppose that $\{Y_i(u)\}_{u\in\mathbb{Z}}$ is obtained via a SAD. Then there exists a $v\in\mathbb{Z}$ such that $\cdots\leq Y_i(v-2)\leq Y_i(v-1)\leq Y_i(v)\geq Y_i(v+1)\geq Y_i(v+2)\geq\cdots$. In other words, the configuration $\{Y_i(u)\}_{u\in\mathbb{Z}}$ is unimodal and v is the mode.

Lemma 3. (*mode height*)
For any $u\in\mathbb{Z}$, $\sup_{i\geq 0} Y_i(u)=1/(|u|+1)$.

Now fix any time $t>0$ and consider the Deffuant model on $\mathbb{Z}$. It is easy to see that there exists a finite interval $[u_a,u_b]\subseteq\mathbb{Z}$ containing 0 such that the Poisson events on the two edges $\{u_a-1,u_a\}$ and $\{u_b,u_b+1\}$ have not happened up to time t (recall that there is a "Poisson clock" on each edge of $\mathbb{Z}$) [16]. Denote by N the number of opinion adjustments occur in $[u_a,u_b]$ up to time t. We arrange the times of these events in the chronological order as

$$0:=\tau_{N+1}<\tau_N<\tau_{N-1}<\cdots<\tau_1\leq t.$$

For $i=1,\cdots,N$, we set u_i be the left end node of the edge $\{u_i,u_i+1\}$ for which u_i and u_i+1 adjust opinions at time τ_i. Given the sequence $u_1,\cdots,u_N$ and $\mu\in(0,1/2]$, we obtain a SAD procedure $\{Y_i(u)\}_{u\in\mathbb{Z}}$. The following proposition is instrumental in understanding the Deffuant dynamics by establishing a link between $\{X_t(u)\}_{u\in\mathbb{Z}}$ and $\{Y_i(u)\}_{u\in\mathbb{Z}}$.

Proposition 1.
For $i=0,1,\cdots,N$,

$$X_t(0)=\sum_{u\in\mathbb{Z}}Y_i(u)X_{\tau_{i+1}}(u).$$
$$\textit{In particular},\ X_t(0)=\sum_{u\in\mathbb{Z}}Y_N(u)X_0(u):=\sum_{u\in\mathbb{Z}}Y_t(u)X_0(u).$$

In other words, the SAD procedure is linked to the Deffuant model in a simple and linear way. The opinion at the origin of any time t can be expressed as a weighted combination of initial opinions across the real line with weights being the above designed SAD. Proposition 1 can be proved straightforwardly by induction over i exactly as in [[16], Lemma 3.1] based on the above construction, since the initial opinion distribution plays no role in the decomposition.

2.2. Flat Points

Another key ingredient toward the solution of Deffuant dynamics is the notion of flat points proposed in [16]. A counterpart in the proof of Lanchier is the set Ω_j [15]. Given $\varepsilon>0$ and the initial configuration $\{X_0(v)\}_{v\in\mathbb{Z}}$ with finite mean (i.e., $\mathrm{E}|X|<\infty$), $u\in\mathbb{Z}$ is said to be an ε-flat point to the right if for all $n\geq 0$,

DOI 10.1002/cplx

$$\frac{1}{n+1}\sum_{v=u}^{u+n} X_0(v) \in [\mathrm{E}X-\varepsilon, \mathrm{E}X+\varepsilon].$$

Similarly, $u \in \mathbb{Z}$ is said to be an ε-flat point to the left if for all $n \geq 0$,

$$\frac{1}{n+1}\sum_{v=u-n}^{u} X_0(v) \in [\mathrm{E}X-\varepsilon, \mathrm{E}X+\varepsilon],$$

and $u \in \mathbb{Z}$ is said to be two-sidedly ε-flat if for all $n, m \geq 0$,

$$\frac{1}{n+m+1}\sum_{v=u-n}^{u+m} X_0(v) \in [\mathrm{E}X-\varepsilon, \mathrm{E}X+\varepsilon].$$

When u is ε-flat to the right, the Kolmogorov strong law of large numbers implies

$$\mathrm{P}\left(\lim_{n\to\infty}\frac{1}{n+1}\sum_{v=u}^{u+n} X_0(v) = \mathrm{E}X\right) = 1.$$

Reasoning as in [16], Lemma 4.2] and using the translation invariance of the configuration $\{X_0(v)\}_{v\in\mathbb{Z}}$, we have

Lemma 4.
For $\varepsilon > 0$ *and* $u \in \mathbb{Z}$, $\mathrm{P}(u \text{ is } \varepsilon\text{−flat to the right}) = \mathrm{P}(u \text{ is } \varepsilon\text{−flat to the left}) > 0$.

By considering three independent events $\mathcal{A}_1 = \{u-1 \text{ is } \varepsilon\text{−flat to the left}\}$, $\mathcal{A}_2 = \{X_0(u) \in [\mathrm{E}X-\varepsilon, \mathrm{E}X+\varepsilon]\}$ and $\mathcal{A}_3 = \{u+1 \text{ is } \varepsilon\text{−flat to the right}\}$, we further have the following result (see [16, Lemma 4.3])

Lemma 5.
For $\varepsilon > 0$ *and* $u \in \mathbb{Z}$, $\mathrm{P}(u \text{ is two−sidedly } \varepsilon\text{−flat}) > 0$.

3. DEFFUANT MODEL: CONSENSUS FORMATION AND CRITICAL VALUE

In this section, we establish the following main result concerning the first impression and critical confidence bound for the Deffuant model.

Theorem 1.
Consider the Deffuant model on $\mathbb{Z}$ *with parameters* $\mu \in (0, 1/2]$ *and* $d \in \mathbb{R}$. *Suppose that* $\mathrm{E}(X^2) < \infty$ *and define*

$$d_c = \inf\{d : \mathrm{P}(|X-\mathrm{E}X| > d) = 0\}. \tag{6}$$

- *If* $d_c < \infty$ *then* d_c *is the critical confidence bound in the following sense. If* $d < d_c$, *then with probability 1 the limiting value* $X_\infty(u) = \lim_{t\to\infty} X_t(u)$ *exists and* $\{|X_\infty(u) - X_\infty(u+1)|\} \in \{0\} \cup [d, \infty)$ *for every* $u \in \mathbb{Z}$. *If* $d > d_c$, *then with probability 1,* $X_\infty(u) = \lim_{t\to\infty} X_t(u) = \mathrm{E}X$ *for every* $u \in \mathbb{Z}$.
- *If* $d_c = \infty$ *then for any* $d \in \mathbb{R}$, *with probability 1 the limiting value* $X_\infty(u) = \lim_{t\to\infty} X_t(u)$ *exists and* $\{|X_\infty(u) - X_\infty(u+1)|\} \in \{0\} \cup [d, \infty)$ *for every* $u \in \mathbb{Z}$.

This theorem means that d_c (either finite or infinite) is the critical confidence bound such that when $d < d_c$ the limiting configuration is piecewise constant interrupted by jumps of size at least d; when $d > d_c$ the complete consensus is formed and the first impression (i.e., the mean of the initial distribution) is confirmed. We mention that $d_c = \infty$ corresponds to the situation where the initial opinion distribution X has an infinite support. It is intuitively plausible that consensus can not be achieved for any $d \in \mathbb{R}$ since there can be an edge $\{v, v+1\}$ such that $|X_0(v) - X_0(v+1)| > d$.

In the following, we show Theorem 1 in two regimes $d < d_c$ and $d > d_c$, respectively. Since the proof is by and large similar as in [16], we focus on the differences and only sketch the similarities.

3.1. Subcritical Regime: $d < d_c$

In this section, we consider the Deffuant model with $d < d_c$, where d_c is defined as in (6).

First we assume that $d_c < \infty$. Take $\delta = (d_c - d)/2 > 0$. For $u \in \mathbb{Z}$, define the following events

$\mathcal{B}(u) = \{|X_t(u) - X_t(u+1)| > d, \text{ for all } t \geq 0\}$,
$\mathcal{C}_1(u) = \{u-1 \text{ is } \delta\text{−flat to the left}\}$,
$\mathcal{C}_2(u) = \{X_0(u) > \mathrm{E}X + d_c - \delta\}$, $\mathcal{C}_3(u) = \{X_0(u) < \mathrm{E}X - d_c + \delta\}$,
and $\mathcal{C}_4(u) = \{u+1 \text{ is } \delta\text{−flat to the right}\}$.

Proposition 2.
Under the assumption of Theorem 1, if $d < d_c$, *then for any* $u \in \mathbb{Z}\, \mathrm{P}(\mathcal{B}(u)) > 0$.

Proof
For $u \in \mathbb{Z}$, define two events $\mathcal{C}(u) = \mathcal{C}_1(u) \cap \mathcal{C}_2(u) \cap \mathcal{C}_4(u)$ and $\mathcal{C}'(u) = \mathcal{C}_1(u) \cap \mathcal{C}_3(u) \cap \mathcal{C}_4(u)$. It follows from the independence that $\mathrm{P}(\mathcal{C}(u)) = \mathrm{P}(\mathcal{C}_1(u))\,\mathrm{P}(\mathcal{C}_2(u))\,\mathrm{P}(\mathcal{C}_4(u))$ and similarly, $\mathrm{P}(\mathcal{C}'(u)) = \mathrm{P}(\mathcal{C}_1(u))\,\mathrm{P}(\mathcal{C}_3(u))\,\mathrm{P}(\mathcal{C}_4(u))$. By the definition (6) we have either (i) $\mathrm{P}(\mathcal{C}_2(u)) > 0$ or (ii) $\mathrm{P}(\mathcal{C}_3(u)) > 0$.

If (i) holds, using Lemma 4 we know that $\mathrm{P}(\mathcal{C}(u)) > 0$. It suffices to show

$$\mathcal{C}(u) \subseteq \mathcal{B}(u). \tag{7}$$

Assume that $\mathcal{C}(u)$ holds. Let $T < \infty$ be the first time that opinion adjustment occurs across either of the edges $\{u-1, u\}$ or $\{u, u+1\}$. Therefore, $X_t(u) = X_0(u)$ for any $t < T$. We will show that such a T does not exist at all.

DOI 10.1002/cplx

Indeed, on one hand there must exist some $t_0 < T$ such that either $X_{t_0}(u-1)$ or $X_{t_0}(u+1)$ exceeds $(\mathrm{E}X+d_c-\delta)-d=\mathrm{E}X+\delta$. Moreover, for any $t<T$ we obtain from Proposition 1 (by replacing 0 with $u+1$ due to translation invariance)

$$X_t(u+1)=\sum_{v\in\mathbb{Z}} Y_t(v)X_0(v),$$

and $Y_t(v)=0$ for all $v\le u$. By Lemma 1 we have

$$Y_t(u+1)\ge Y_t(u+2)\ge\cdots\ge Y_t(u+N)>0=Y_t(u+N+1)=\cdots$$

for some $1\le N<\infty$. Set $c_k=k(Y_t(u+k)-Y_t(u+k+1))\ge 0$ for $k=1,\cdots,N$. Calculating as in [16, Eqs. (19) and (20)] gives $\sum_{n=1}^N c_n=1$ and

$$X_t(u+1)=\sum_{n=1}^N c_n\left(\frac{1}{n}\sum_{k=1}^n X_0(u+k)\right). \tag{8}$$

Since the event $\mathcal{C}_4(u)$ holds, we see from (8) that

$$X_t(u+1)\in[\mathrm{E}X-\delta,\mathrm{E}X+\delta]. \tag{9}$$

Analogously, we can show that $X_t(u-1)\in[\mathrm{E}X-\delta,\mathrm{E}X+\delta]$. This fact contradicts with the existence of such a $t_0<T$ above. Therefore, we must have $T=\infty$.
Furthermore, noting that $\mathcal{C}_2(u)$ holds and using (9) we obtain

$$|X_t(u)-X_t(u+1)|>\mathrm{E}X+d_c-\delta-(\mathrm{E}X+\delta)=d_c-2\delta=d$$

for all $t\ge 0$. Hence, $\mathcal{B}(u)$ holds, and (7) is established.
If (ii) holds, using Lemma 4 likewise we know that $\mathrm{P}(\mathcal{C}'(u))>0$. It suffices to prove

$$\mathcal{C}'(u)\subseteq\mathcal{B}(u). \tag{10}$$

This can be shown analogously as in case (i), which concludes the proof.
Notice that the above proof essentially indicates that, for all $u\in\mathbb{Z}$, $\mathcal{C}(u)\cup\mathcal{C}'(u)\subseteq\mathcal{B}(u)$ and $\mathrm{P}(\mathcal{B}(u))\ge\mathrm{P}(\mathcal{C}(u)\cup\mathcal{C}'(u))>0$. By the ergodicity ([17, p. 340 Theorem (1.3)]) of the indicator processes $\{I_{\mathcal{B}(u)}\}_{u\in\mathbb{Z}}$ and $\{I_{\mathcal{C}(u)\cup\mathcal{C}'(u)}\}_{u\in\mathbb{Z}}$, we can obtain the following corollary (c.f. [[16], Lemma 5.2])

Corollary 1.
With probability 1, there are infinitely many nodes u to the left (and right) of 0 such that $\mathcal{B}(u)$ happens. The same thing holds for $\mathcal{C}(u)\cup\mathcal{C}'(u)$.

Proposition 3.
Under the assumption of Theorem 1, if $d<d_c$, then with probability 1 the limiting value $X_\infty(u)=\lim_{t\to\infty}X_t(u)$ exists and $\{|X_\infty(u)-X_\infty(u+1)|\}\in\{0\}\cup[d,\infty)$ for every $u\in\mathbb{Z}$.

Proof
Given the initial opinion configuration $\{X_0(u)\}_{u\in\mathbb{Z}}$, let u_1 be a node such that $\mathcal{C}(u_1-1)\cup\mathcal{C}'(u_1-1)$ happens, and let $u_2=\min\{u>u_1:\mathcal{C}(u)\cup\mathcal{C}'(u)\text{ happens}\}$. Since $\mathcal{C}(u)\cup\mathcal{C}'(u)\subseteq\mathcal{B}(u)$ for every $u\in\mathbb{Z}$, the opinions in the interval $\{u_1,u_1+1,\cdots,u_2\}$ will not be affected by nodes outside and vice versa. From Corollary 1, we know that each $u\in\mathbb{Z}$ must sit in some such interval. Hence, we only need to show the proposition for every $u\in\{u_1,u_1+1,\cdots,u_2\}$.
The rest of the proof essentially follows from Theorem 5.3 [16]. We outline the argument as follows. Define the energy of the interval $\{u_1,u_1+1,\cdots,u_2\}$ at time t as

$$W_t=\sum_{u\in\{u_1,u_1+1,\cdots,u_2\}} X_t(u)^2\ge 0.$$

If the nodes u and $u+1$ in the interval exchange opinions, W_t drops by an amount of $2\mu(1-\mu)|X_{t-}(u)-X_{t-}(u+1)|$, and W_t is always decreasing with respect to time t. This fact together with the conditional version of the Borel-Cantelli Lemma [17, p. 240, Corollary (3.2)] can be used to show

$$\lim_{t\to\infty}\max\{|X_t(u)-X_t(u+1)|I_{\{|X_t(u)-X_t(u+1)|\le d\}}:u\in\{u_1,u_1+1,\cdots,u_2-1\}\}=0. \tag{11}$$

For any edge $\{u,u+1\}$ in the interval $\{u_1,u_1+1,\cdots,u_2\}$, a single opinion adjustment can only increase $|X_t(u)-X_t(u+1)|$ by at most μd. Exploiting (11) we can see that either $|X_t(u)-X_t(u+1)|>d$ for all large enough t or $\lim_{t\to\infty}|X_t(u)-X_t(u+1)|=0$. Finally, this together with the fact that the quantity $\sum_{u\in\{u_1,u_1+1,\cdots,u_2\}}X_t(u)$ remains unchanged over time can be used to show the existence of the limit $\lim_{t\to\infty}X_t(u)$.
If $d_c=\infty$, it is easy to check that all the arguments in this section still hold true by replacing d_c with $d+\varepsilon$ for any $\varepsilon>0$. Hence, the subcritical part of Theorem 1 is completed.

3.2. Supercritical Regime: $d>d_c$

To understand the behavior of the Deffuant model in the regime $d>d_c$ (with $d_c<\infty$) the following two lemmas are critical.

Lemma 6.
Suppose that the assumption of Theorem 1 holds. Fix $d\in\mathbb{R}$. With probability 1, for any $u\in\mathbb{Z}$, either $|X_t(u)-X_t(u+1)|>d$ for all large enough t or $\lim_{t\to\infty}|X_t(u)-X_t(u+1)|=0$.

Proof
For each $u\in\mathbb{Z}$, similarly as in Proposition 3, we define the energy at node u as $W_t(u)=X_t(u)^2$. We further define a continuous-time step function $W_t^\dagger(u)$ by $W_0^\dagger(u)=0$ and

DOI 10.1002/cplx

$W_t^{\dagger}(u)$ jumps an amount of $2\mu(1-\mu)|X_{t-}(u)-X_{t-}(u+1)|^2$ at time t when opinion adjustment occurs on the edge $\{u,u+1\}$ [16]. We can show as in Lemma 6.2 [16] that for any $u\in\mathbb{Z}$ and $t\geq 0$,

$$\mathrm{E}\,W_t(u)+\mathrm{E}\,W_t^{\dagger}(u)=\mathrm{E}\,(X^2)<\infty.$$

Using this fact and the conditional Borel-Cantelli Lemma, we can finish the proof as in Proposition 3 (see the proof of Proposition 6.1 [16]).

Using the powerful tool linking SAD and the Deffuant model (Proposition 1), we have the following result, which can be shown verbatim follows the proof of Lemma 6.3 [16]. The proof entails combining an argument similar as Proposition 2 and a discussion for the location of the mode on $\mathbb{Z}$ (see Lemma 2).

Lemma 7

Given an initial configuration $\{X_0(u)\}_{u\in\mathbb{Z}}$ and $\varepsilon>0$. If $u\in\mathbb{Z}$ is two-sidedly ε-flat, then $X_t(u)\in[\mathrm{E}X-6\varepsilon,\mathrm{E}X+6\varepsilon]$ for all $t\geq 0$.

Proposition 4

Under the assumption of Theorem 1, if $d>d_c$, then with probability 1 $X_\infty(u)=\lim_{t\to\infty}X_t(u)=\mathrm{E}X$ for every $u\in\mathbb{Z}$.

Proof

Take an $\varepsilon>0$ satisfying $d>d_c+6\varepsilon$. We first show that with probability 1

$$\lim_{t\to\infty}|X_t(u)-X_t(u+1)|=0 \tag{12}$$

for any $u\in\mathbb{Z}$ (see Proposition 6.4 [16]). By Lemma 6, it suffices to show that for each u,

$$\mathrm{P}(|X_t(u)-X_t(u+1)|>d \text{ for all large enough } t)=0. \tag{13}$$

Suppose that the probability in (13) is positive. Then the event in (13) happens for infinitely many v with probability 1 by using the ergodicity theorem. Arguing as in Proposition 3, we obtain that the limit $X_\infty(u)$ exists and $\{|X_\infty(u)-X_\infty(u+1)|\}\in\{0\}\cup[d,\infty)$ for every $u\in\mathbb{Z}$. Lemma 5 implies that, with probability 1, there exists a node w which is two-sidedly ε-flat. From Lemma 7, we know that $X_\infty(w)\in[\mathrm{E}X-6\varepsilon,\mathrm{E}X+6\varepsilon]$. By Lemma 6, $X_\infty(w+1)$ must either exceed $X_\infty(w)$ by at least d, or fall below $X_\infty(w)$ by at least d, or equal $X_\infty(w)$. In other words, by the choice of ε, we have either (i) $X_\infty(w+1)>\mathrm{E}X+d_c$, or (ii) $X_\infty(w+1)<\mathrm{E}X-d_c$, or (iii) $X_\infty(w+1)=X_\infty(w)$. By the definition (6) and the fact that $\min_{u\in\mathbb{Z}}X_0(u)\leq X_\infty(w+1)\leq\max_{u\in\mathbb{Z}}X_0(u)$, the cases (i) and (ii) are impossible. Accordingly, we can show $X_\infty(w)=X_\infty(u)$ iteratively for all $u\in\mathbb{Z}$. This, however, contradicts the infinitely many v(13) is established and (12) holds.

Next, for the node w obtained above, we have $X_t(w)\in[\mathrm{E}X-6\varepsilon,\mathrm{E}X+6\varepsilon]$ for all $t\geq 0$. For any $u\in\mathbb{Z}$, we obtain with probability 1 that $X_t(u)\in[\mathrm{E}X-7\varepsilon,\mathrm{E}X+7\varepsilon]$ for large enough t by invoking (12) (since there are only finitely many edges between u and w). Taking $\varepsilon\to 0$ completes the proof.

4. SIMULATION STUDY

4.1. Methodology

In order to demonstrate and deepen our theoretical results, we carry out the simulations on rings of different sizes n, where each node is connected to its two neighbors on either side. A real life parallel can be seen with residents in modern large apartment buildings where people tend to build walls of privacy in an intellectual/emotional sense and only know their neighbors live right next door [18]. We propose several initial opinion distributions which are modeled by continuous probability distributions. To be specific, we consider the following four classes of probability distributions (see Figure 1 for their density curves):

- $\mathrm{U}(a,b)$: uniform distribution on the interval $[a, b]$, whose probability density function is $f_{\mathrm{U}(a,b)}(x)=\frac{1}{b-a}I_{\{a\leq x\leq b\}}$. We interpret it as even opinions.
- $\mathrm{Beta}(\alpha,\beta)$: beta distribution on the interval [0, 1] with parameters $\alpha>0$ and $\beta>0$. Its probability density function is $f_{\mathrm{Beta}(\alpha,\beta)}(x)=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}I_{\{0\leq x\leq 1\}}$, where B is the beta function. We will consider three different pairs of parameters, namely, $(\alpha,\beta)=(0.3,0.3)$ representing polarized opinions; $(\alpha,\beta)=(3,3)$ representing unbiased unimodal opinions; and $(\alpha,\beta)=(2,5)$ representing biased unimodal opinions.
- $\mathrm{PL}(x_0,\gamma)$: power-law (Pareto) distribution on $[x_0,\infty)$ with parameter $x_0>0$ and $\gamma>0$. Its probability density function is $f_{\mathrm{PL}(x_0,\gamma)}(x)=\gamma x_0^{\gamma}x^{-\gamma-1}I_{\{x\geq x_0\}}$. We interpret it as divergent biased opinions.
- $\mathrm{N}(\hat{\mu},\hat{\sigma}^2)$: normal (or Gaussian) distribution on $\mathbb{R}$, whose probability density function is $f_{\mathrm{N}(\hat{\mu},\hat{\sigma}^2)}(x)=\frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\exp\left(-\frac{(x-\hat{\mu})^2}{2\hat{\sigma}^2}\right)$. It represents divergent unbiased unimodal opinions.

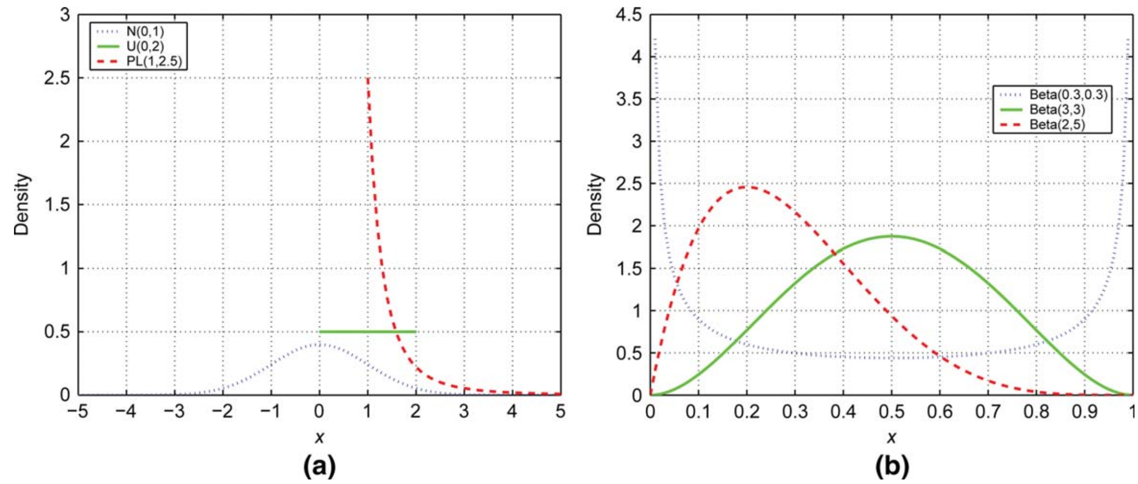
Table 1 summarizes some important properties regarding our theoretical result (Theorem 1) for these distributions.

4.2. First Impression Effect

In this section, the simulations have been made to display the time evolution of opinions among a population of $n=500$ individuals on a ring; see Figure 2. We deal with six typical initial opinion distributions, namely

DOI 10.1002/cplx

FIGURE 1



Depiction of probability density functions of initial opinions studied in the simulations. (a) uniform distribution U(a, b) with $a = 0, b = 2$ (green solid line), power-law distribution PL(x_0, γ) with $x_0=1, \gamma=2.5$ (red dashed line), and normal distribution N$(\hat{\mu}, \hat{\sigma}^2)$ with $\hat{\mu}=0, \hat{\sigma}=1$ (blue dotted line); (b) beta distributions Beta(α, β) with $(\alpha, \beta)=(0.3, 0.3)$ (blue dotted line), $(\alpha, \beta)=(3, 3)$ (green solid line), and $(\alpha, \beta)=(2, 5)$ (red dashed line).

$\mathrm{U}(0,2), \mathrm{Beta}(0.3, 0.3), \mathrm{Beta}(3,3), \mathrm{Beta}(2,5), \mathrm{PL}(1, 2.5)$, and N(0,1), as described above. The parameters we used in the simulations are summarized in Table 2.

The straightforward method for illustrating the evolution of opinions is to consider continuous-time evolution by monitoring each Poisson jumps. However, the convergence in our situation is much slower than that in a fully mixed population [6,19] and the machine incurs out-of-memory error due to the overwhelming computation. Since our main goal here is to confirm the first impression effect, we plot in Figure 2 the opinion evolutions by compressing (and discretizing) time axis. Specifically, each time unit in Figure 2(a–f) corresponds to 50,000 times of Poisson events. In the insets of Figure 2(a–f), we perform independent simulations with each time unit corresponding to 1,000 times of Poisson events, respectively.

We observe from Figure 2 the following. First, the opinions converge to the average EX of initial opinions for all the six situations. For the first four cases in Figure 2(a–d), the first impression effect predicted in Theorem 1 is confirmed since we take $d > d_c$ (c.f. Table 2). For the last two cases in Figure 2(e–f), we see that final consensus are also reached at the average EX. We performed a number of tests by using different confidence bound d (taking $d > d_c$ when $d_c < \infty$, and taking d large enough when $d_c = \infty$) and different size n of nodes varying from 500 to 1000, and they confirm the first impression effect.

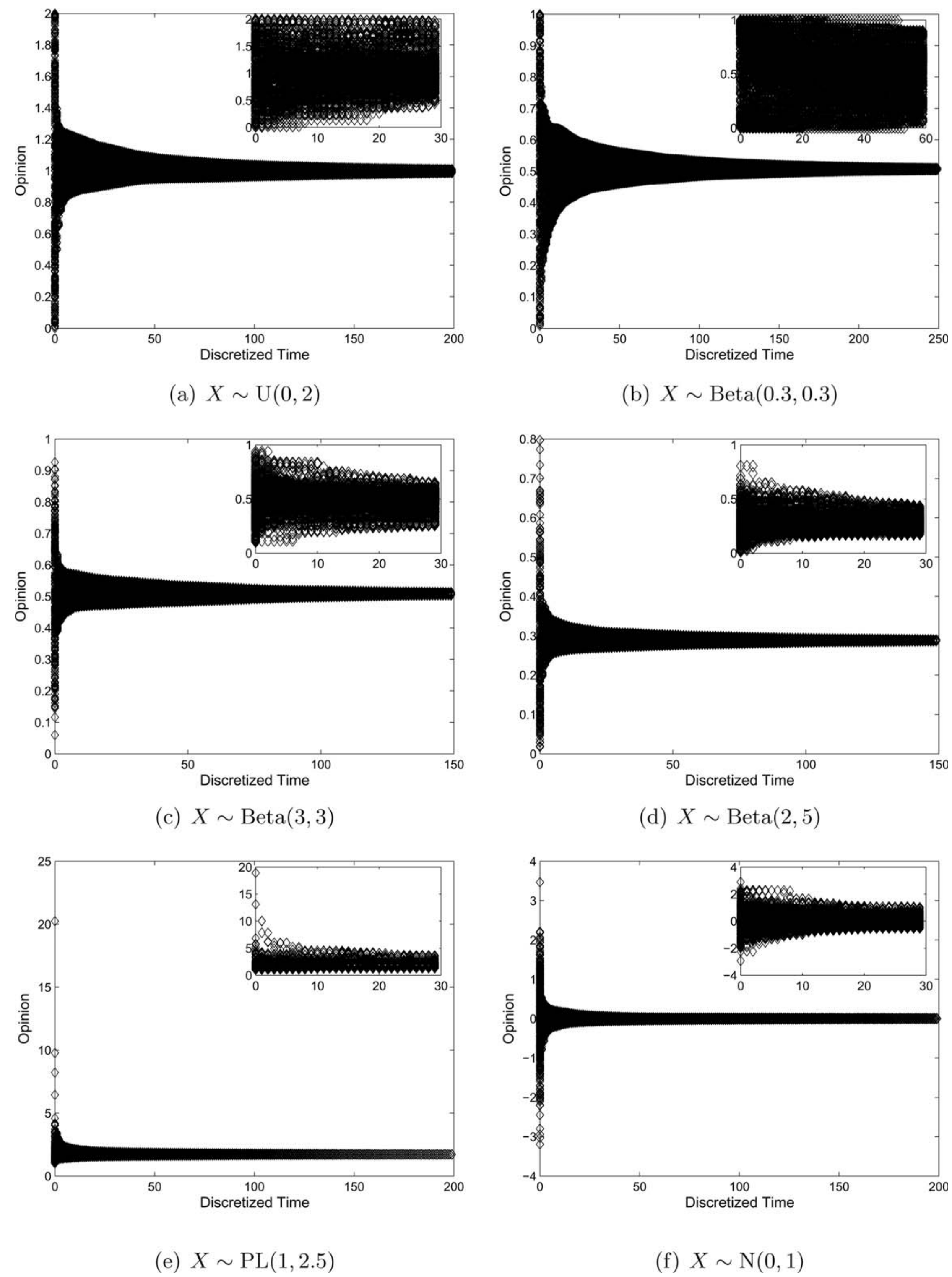
Second, scattered opinion distributions (such as U(0, 2)) and B(0.3, 0.3) converge much slower than unimodal opinion distributions (such as B(3, 3), B(2, 5) and N(0, 1)), especially at early times. For example, in Figure 2(b) the opinion discrepancy is around [0.3, 0.7] at $t \approx 10$ while it is around [0.4, 0.6] in Figure 2(c). These differences can also be seen clearly from the insets: there are no evident convergence until time $t = 60$ for Figure 2(b) inset, but the convergence can be discerned as early as $t \approx 30$ for Figure 2(c,d) insets. For opinion distributions with divergent supports [such as PL(1, 2.5) and N(0, 1)] the consensus becomes sensitively contingent on d. When d is large, fast consensus can be expected (as is shown in Figure 2(e,f), where we choose d almost equal to the maximal initial opinion difference). The reason why for PL(1, 2.5) we need a much larger d than other cases (c.f. Table 2) is that power-law distribution has a heavy tail, and "outliers" are more likely to appear. For example, if we take $d = 10$ in Figure 2(e), we probably can not get consensus since there is an opinion at around 20 and the only second largest opinion is at around 10. The rate of consensus for different opinion distributions will be further studied in section 4.4 by rescaling opinions on the same range and fixing both μ and d.

4.3. Critical Confidence Bound

To determine the critical confidence bound d_c, we resort to Monte Carlo simulations. We fix μ=0.5 as above since it does not affect the final configuration. Given the number n of nodes, confidence bound d and the initial opinion distribution X, we conduct the Deffuant model algorithm on 1000 samples. The Deffuant algorithm proceeds until no node changes its opinion by more than 0.0001 for 100,000 times of consecutive Poisson events. Let P_c be the fraction of samples which reach a complete

DOI 10.1002/cplx

FIGURE 2



Evolution of opinions with $n = 500$ and μ=0.5. Each opinion is represented by a hollow diamond. In the main pictures, one time unit corresponds to 50,000 times of Poisson events; while in the insets one time unit corresponds to 1000 times of Poisson events. In each subfigure, the inset displays another independent simulation with respect to the main picture.

TABLE 2

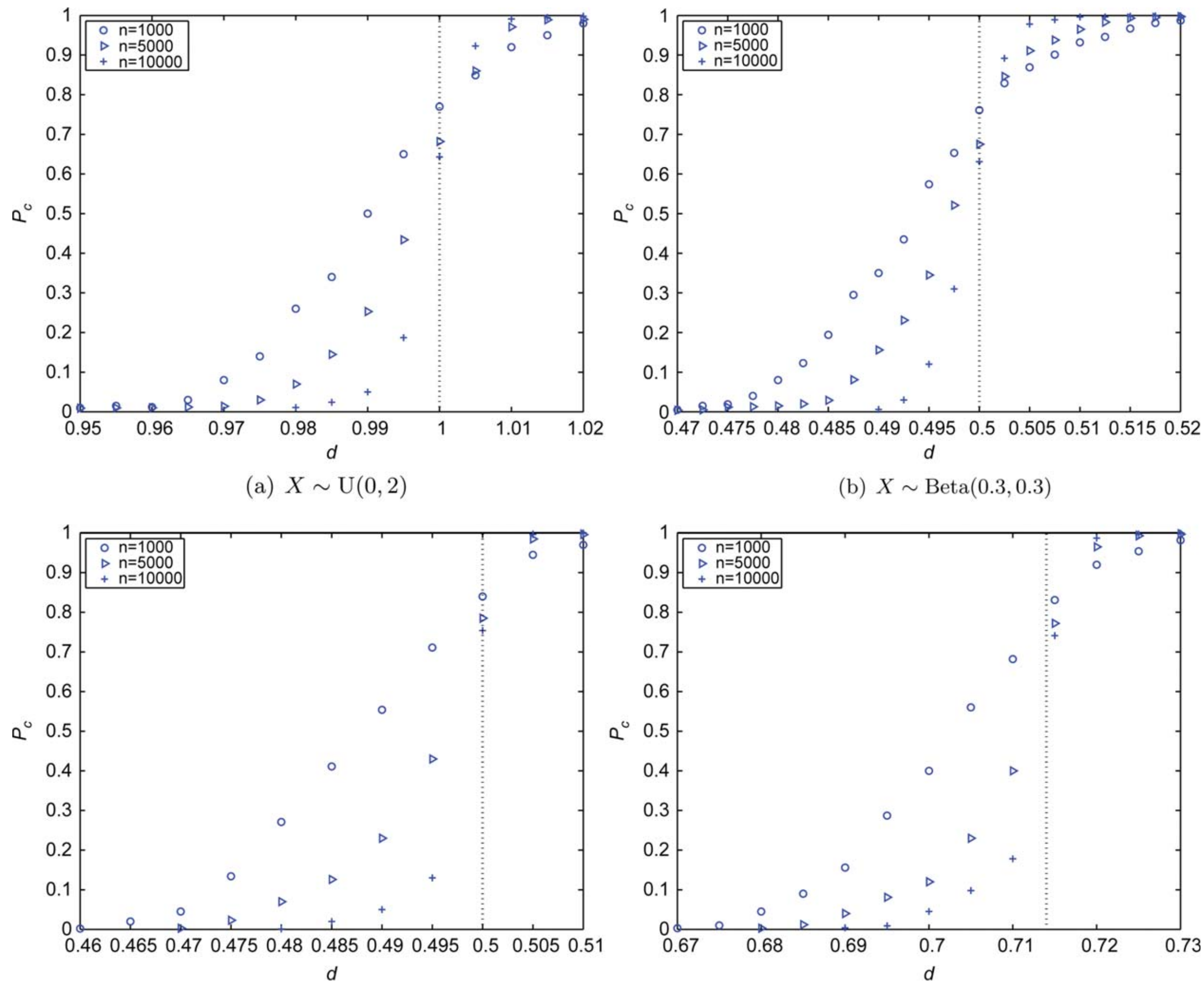
The Parameters Used in the Simulations Corresponding to Figure 2

X	EX	d_c	d	n	μ	Fig. no.
U(0, 2)	1	1	1.2	500	0.5	2(a)
Beta(0.3, 0.3)	0.5	0.5	0.6	500	0.5	2(b)
Beta(3, 3)	0.5	0.5	0.6	500	0.5	2(c)
Beta(2, 5)	0.286	0.714	0.8	500	0.5	2(d)
PL(1, 2.5)	1.67	∞	20	500	0.5	2(e)
N(0, 1)	0	∞	4	500	0.5	2(f)

consensus. In Figure 3, we plot P_c as a function of confidence bound d for four initial opinion distributions U(0, 2), Beta(0.3, 0.3), Beta(3, 3), and Beta(2, 5).

From Figure 3(a–d), we clearly observe that P_c increases for $d < d_c$ and then saturates to 1 for $d > d_c$ in each of the four cases. By examining the growth of P_c against different population size n, we may conclude that P_c will converge to a step function in the limit $n \to \infty$, implying a critical confidence bound d_c. This agrees with our Theorem 1. Our results are compatible with an early study of the critical bound [9], where individuals sit on the sites of a square lattice and random graphs. Finally, we remark that d_c can not be finite for those with divergent opinion supports [such as PL(1, 2.5) and N(0, 1)],

FIGURE 3

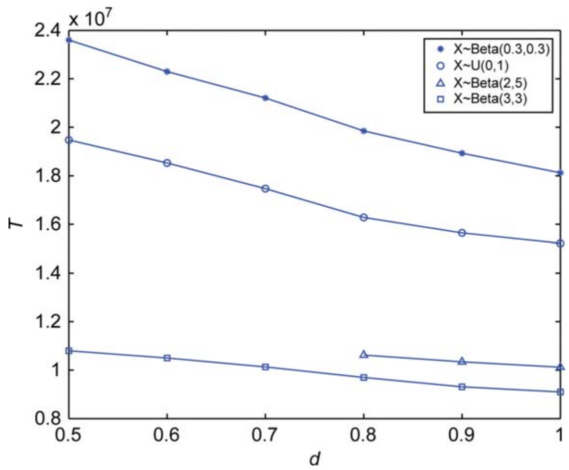


(a) $X \sim \mathrm{U}(0, 2)$ (b) $X \sim \mathrm{Beta}(0.3, 0.3)$ (c) $X \sim \mathrm{Beta}(3, 3)$ (d) $X \sim \mathrm{Beta}(2, 5)$

Fraction of samples with complete opinion consensus as a function of the confidence bound. Three different numbers of individuals located on rings are studied: n = 1000, 5000, and 10,000. The critical confidence bound d_c is indicated by a vertical dotted line in each subfigure (c.f. Table 2).

DOI 10.1002/cplx

FIGURE 4



Time of reaching complete consensus for different initial opinion distributions as a function of confidence bound d. The results for Beta(2, 5) starts from $d = 0.8$ since we only consider the supercritical regime $d > d_c$.

since it is always possible (i.e., with positive probability) to find two nodes with distance arbitrarily large for large enough n.

4.4. Comparison of Consensus Rate

In this section, we explore the rate of consensus in the Deffuant model by considering four initial opinion distributions U(0, 1), Beta(0.3, 0.3), Beta(3, 3), and Beta(2, 5). Opinions with these distributions lie in the same range [0, 1]. Set $\Delta(t)=\max_{u,v}\{X_t(u)-X_t(v)\}$ and

$$T=T_X=\min\{t:\Delta(t)<0.0001\}.$$

We refer to T as the time of reaching consensus. Simulations are done for a population of $n = 1000$ individuals on a ring and μ=0.5. Figure 4 represents the rate of consensus versus confidence bound d. We observe the following. First, for each given d, the times of reaching consensus are arranged decreasingly as

$$T_{\text{Beta}(0.3,0.3)} > T_{\text{U}(0,1)} > T_{\text{Beta}(2,5)} > T_{\text{Beta}(3,3)}.$$

This relation suggests that

$$\begin{aligned}&\text{polarized opinions} \prec \text{even opinions}\\ &\prec \text{unimodal (but biased) opinions}\\ &\quad \prec \text{unimoal and unbiased opinions},\end{aligned}$$

where "$\prec$" means "converges slower than". This agrees with the observation in [14]. Furthermore, we see that the difference between $T_{\text{Beta}(3,3)}$ and $T_{\text{Beta}(2,5)}$ is relatively small. Note that there is absent of mainstream view in an even opinion configuration and there are (at least) two mainstream views in a polarized opinion configuration. Hence, we may conclude that unimodality (the emergence of a single mainstream view) is a prominent feature which contributes to fast consensus of opinions.

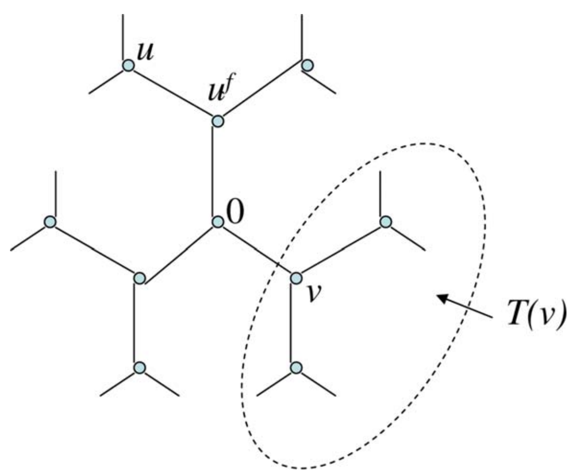
Second, for each of these distributions convergence becomes faster for larger d. This is intuitively clear since when d is rising, more nodes are able to adjust their opinions with each other.

Third, the convergence time T is more sensitive to U(0, 1) and Beta(0.3, 0.3) than Beta(3, 3). For example, T reduces by almost 25% for Beta(0.3, 0.3) but around 15% for Beta(3, 3) when d increases all the way from 0.5 to 1. This can be explained as follows. Even and polarized opinions [represented by, e.g., U(0, 1) and Beta(0.3, 0.3)] are essentially scattered, and the increase of d plays a more important role in compromising the opinions since the unimodal opinions [represented by, e.g., Beta(3, 3) and Beta(2, 5)] are aggregated to some extent at the outset.

5. DISCUSSION

We have shown that the Deffuant model on $\mathbb{Z}$ with initial opinion distribution X exhibits a critical confidence bound d_c provided the second order moment $\text{E}X < \infty$. When $d > d_c$, the opinions will converge eventually with probability one to the initial mean value EX, a first impression phenomenon. When $d < d_c$, the limit configuration is piecewise constant interrupted by jumps of size at least d. It would be very interesting to better

FIGURE 5



Depiction of an infinite Cayley tree $\mathbb{CT}_r$ with $r = 3$. The root node is labeled 0. Each node u has a unique father node u^f. For any node v, $T(v)$ is the subtree with root v.

DOI 10.1002/cplx

understand the first impression effect of the Deffuant model on general networks. An intermediate step toward this goal can be an infinite Cayley tree (see e.g., [20,21]). To formalize the question precisely we need the following definitions.

For $r \in \mathbb{N}$, let $\mathbb{CT}_r$ be a labeled r-regular infinite tree without leaf nodes; see Figure 5. The root node is labeled 0. Denote by $T(u)$ the subtree with root $u \in \mathbb{CT}_r$. Hence, $T(0)=\mathbb{CT}_r$. For any $0 \neq u \in \mathbb{CT}_r$, there is a unique father node, denoted by u^f. Thus each edge in $\mathbb{CT}_r$ has a unique representation like $\{u, u^f\}$. Analogous to (5), given a sequence $u_1, u_2, \cdots \in \mathbb{CT}_r$ and $\mu \in (0, 1/2]$, the associated SAD process $\{Y_i(u)\}_{u \in \mathbb{CT}_r}$ can be defined as

$$Y_i(u)=\begin{cases} Y_{i-1}(u)+\mu(Y_{i-1}(u^f)-Y_{i-1}(u)) & \text{for } u=u_i; \\ Y_{i-1}(u)+\mu(Y_{i-1}(u_i)-Y_{i-1}(u)) & \text{for } u=u_i^f; \\ Y_{i-1}(u) & \text{for } u \in \mathbb{CT}_r \backslash \{u_i, u_i^f\}. \end{cases}$$

If $\{u_1 \leftrightarrow u_k\} := \{u_1, u_2, \cdots, u_k\}$ is a path between nodes u_1 and u_k on $\mathbb{CT}_r$, and $Z : \mathbb{CT}_r \to \mathbb{R}$, then we write

$$Z(u_1) \geq \cdots \geq Z(u_k)$$

meaning that $Z(u_1) \geq Z(u_2) \geq \cdots \geq Z(u_k)$. Since the path between u_1 and u_k is unique, the above notations are well-defined.

Note that when $r \geq 3$, the monotonicity (c.f. Lemma 1) no longer holds, since liquids can pass through different branches along the tree and much more complicated phenomenon may happen. However, we are able to establish the following "constrained" monotonicity result.

Proposition 5
Suppose that $\{Y_i(u)\}_{u \in \mathbb{CT}_r}$ *is obtained via a SAD such that* $u_j \in \{0 \leftrightarrow v\} \cup T(v)$ *for all* $1 \leq j \leq i$. *Then*

$$Y_i(0) \geq \cdots \geq Y_i(v^f) \geq \max_{u \in T(v)} Y_i(u). \tag{14}$$

Proof
Assume that (14) holds for $i = k - 1$, and we need to show it holds for $i = k$. It suffices to prove the following three cases:

a. If $u_k \in \{0 \leftrightarrow (v^f)^f\}$, $Y_k\left(\left(u_k^f\right)^f\right) \geq Y_k\left(u_k^f\right) \geq Y_k(u_k) \geq Y_k(w)$, where $w^f = u_k$;
b. If $u_k = v^f$, $Y_k\left(\left(u_k^f\right)^f\right) \geq Y_k\left(u_k^f\right) \geq Y_k(u_k) \geq \max_{u \in T(v)} Y_k(u)$;
c. If $u_k = v$, $Y_k\left(\left(u_k^f\right)^f\right) \geq Y_k\left(u_k^f\right) \geq \max_{u \in T(v)} Y_k(u)$.
To see (a) holds, we note that

$$Y_k\left(\left(u_k^f\right)^f\right) = Y_{k-1}\left(\left(u_k^f\right)^f\right) \geq (1-\mu)Y_{k-1}\left(u_k^f\right)+\mu Y_{k-1}(u_k) = Y_k\left(u_k^f\right),$$
$$Y_k\left(u_k^f\right) = Y_{k-1}\left(u_k^f\right)+\mu\left(Y_{k-1}(u_k)-Y_{k-1}\left(u_k^f\right)\right)$$
$$\geq Y_{k-1}\left(u_k^f\right)+(1-\mu)\left(Y_{k-1}(u_k)-Y_{k-1}\left(u_k^f\right)\right) = Y_k(u_k),$$

and

$$Y_k(w) = Y_{k-1}(w) \leq (1-\mu)Y_{k-1}(u_k)+\mu Y_{k-1}\left(u_k^f\right) = Y_k(u_k).$$

To see (b) holds, we note that the first two inequalities can be shown similarly as in case (a). The last inequality holds since

$$\max_{u \in T(v)} Y_k(u) = \max_{u \in T(v)} Y_{k-1}(u) \leq (1-\mu)Y_{k-1}(u_k)+\mu Y_{k-1}\left(u_k^f\right) = Y_k(u_k).$$

To see (c) holds, we note that the first inequality can be shown similarly as above. Since $Y_{k-1}\left(u_k^f\right) \geq Y_{k-1}(u_k)$, we have $Y_k\left(u_k^f\right) \geq Y_k(u_k)$. Moreover, we obtain

$$\max_{u \in T(v) \backslash \{u_k\}} Y_k(u) = \max_{u \in T(v) \backslash \{u_k\}} Y_{k-1}(u) \leq (1-\mu)Y_{k-1}\left(u_k^f\right)+\mu Y_{k-1}\left(\left(u_k^f\right)^f\right) = Y_k\left(u_k^f\right)$$

Therefore, the second inequality also holds.
Let $\ell(u)$ be the distance of u to the root 0. We have the following result similarly as Lemma 3 (see [16, Theorem 2.3]).

Proposition 6
For any $u \in \mathbb{CT}_r$, $\sup_{i \geq 0} Y_i(u) = 1/(\ell(u)+1)$.
It is an intriguing open problem to get a suitable characterization of "flat points" on $\mathbb{CT}_r$ so that the critical value of confidence bound can be established. We are currently working on the related issues and will release the results elsewhere.

References

1. Clifford, P.; Sudbury, A. A model for spatial conflict. Biometrika 1973, 60, 581–588.
2. Galam, S. Minority opinion spreading in random geometry. Eur Phys J B 2002, 25, 403–406.
3. Nowak, A.; Szamrej, J.; Latané, B. From private attitude to public-opinion: A dynamic theory of social impact. Psychol Rev 1990, 97, 362–376.
4. Sznajd-Weron, K.; Sznajd, J. Opinion evolution in closed community. Int J Mod Phys C 2000, 11, 1157–1165.
5. Deffuant, G.; Neau, D.; Amblard, F.; Weisbuch, G. Mixing beliefs among interacting agents. Adv Complex Syst 2000, 3, 87–98.
6. Hegselmann, R.; Krause, U. Opinion dynamics and bounded confidence: Models, analysis and simulation. J Artif Soc Soc Simul 2002, 5, 2.
7. Castellano, C.; Fortunato, S.; Loreto, V. Statistical physics of social dynamics. Rev Mod Phys 2009, 81, 591–646.

DOI 10.1002/cplx

8. Lorenz, J. Continuous opinion dynamics under bounded confidence: A survey. Int J Mod Phys C 2007, 18, 1819–1838.
9. Fortunato, S. Universality of the threshold for complete consensus for the opinion dynamics of Deffuant et al. Int J Mod Phys C 2004, 15, 1301–1307.
10. Lorenz, J.; Urbig, D. About the power to enforce and prevent consensus by manipulating communication rules. Adv Complex Syst 2007, 10, 251–269.
11. Laguna, M.F.; Abramson, G.; Zanette, D.H. Minorities in a model for opinion formation. Complexity 2004, 9, 31–36.
12. Laslier, J.F. Diffusion d'information et évaluations séquentielles. Econ Appl 1989, 42–3, 155–170.
13. Jacobmeier, D. Focusing of opinions in the Deffuant model: First impression counts. Int J Mod Phys C 2006, 17, 1801–1808.
14. Kou, G.; Zhao, Y.; Peng, Y.; Shi, Y. Multi-level opinion dynamics under bounded confidence. PLoS ONE 2012, 7, e43507.
15. Lanchier, N. The critical value of the Deffuant model equals one half. ALEA Lat Am J Probab Math Stat 2012, 9, 383–402.
16. Häggström, O. A pairwise averaging procedure with application to consensus formation in the Deffuant model. Acta Appl Math 2012, 119, 185–201.
17. Durrett, R. Probability: Theory and Examples; Duxbury Press: Belmont, 1996.
18. Rokach, A. Loneliness then and now: reflections on social and emotional alienation in everyday life. Curr Psychol 2004, 23, 24–40.
19. Ben-Naim, E.; Krapivsky, P.; Redner, S. Bifurcations and patterns in compromise processes. Phys D 2003, 183, 190–204.
20. Godsil, C.; Royle, G. Algebraic Graph Theory; Springer: New York, 2001.
21. Shang, Y. A note on the perturbation of mixed percolation on the hierarchical group. Z Naturforsch A 2013, 68, 475–478.

DOI 10.1002/cplx