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L^1 group consensus of multi-agent systems with switching topologies and stochastic inputs

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ABSTRACT

Understanding how interacting subsystems of an overall system lead to cluster/group consensus is a key issue in the investigation of multi-agent systems. In this Letter, we study the L^1 group consensus problem of discrete-time multi-agent systems with external stochastic inputs. Based on ergodicity theory and matrix analysis, L^1 group consensus criteria are obtained for multi-agent systems with switching topologies. Some numerical examples are provided to illustrate the effectiveness and feasibility of the theoretical results.

1. Introduction

Multi-agent systems arise from different fields in natural and artificial systems, including flocking of birds, schooling of fish [1], and coordination of autonomous agents [2]. Scientifically, how to design local rules by which interacting agents lead to emergence of collective/consensus behavior is a fundamental and challenging problem to be understood. Over the past few decades, the consensus problems of multi-agent systems have attracted increasing attention from researchers in diverse disciplines, such as physics [3–6], biology [7–9], and control theory [10–14].

Consensus or agreement problems generally mean the convergence to a common value asymptotically or in finite time among all members of a group via local interaction rules/protocols. Understanding the collective behavior is one of the important questions in non-equilibrium statistical physics. In 1995, Vicsek et al. [4] first proposed a simple model, where n self-propelled agents moving in the plane with the same constant speed and with the headings of each agent updated according to the averaged direction of its neighbors. Via computer simulation, they showed that the system will synchronize, i.e. each agent behaves as others do in its neighborhood, which characterizes the connection between the non-equilibrium dynamics and the equilibrium phase transitions. Theoretical study was later performed by Jadbabaie et al. [11] for a linearized version of Vicsek's model. It was shown that the system will achieve a consensus if the underlying communication graphs are jointly connected within some contiguous bounded time intervals. Along this line, numerous results and effective algorithms are reported during the last few years. Useful approaches to consensus problems include Lyapunov's direct method [15,16], algebraic graph theory [13,14,17], linear iterations [18], convex analysis [12], and linear matrix inequality [19].

Recently, a major advance towards the consensus problem is made by the endeavor of several authors [20–26], where group (or cluster) consensus is formulated and studied. In a complex network consisting of multiple sub-networks/sub-groups, group consensus means that the agents in each sub-network reach a consistent value asymptotically, and there is no consistent value among different sub-networks. This novel type of consensus is more appealing for complex practical applications, such as the pattern formation in bacteria colonies and the cluster formation of opinions in social networks, since the agreements in these real-life scenarios are often different with the changes of environments, situations or even time. Recent particle-based simulations on multiple-component swarms [27], for example, unravel that the motion of a swarm may lead to respective flocking behaviors from group to group within a flock due to different behavioral parameters involved in the model. Another striking example lies in the social dynamics [5], where Monte Carlo simulations reveal that interacting

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agents in a social network tend to form different opinion clusters, and the number of clusters can be estimated by certain "confidence bound" [28].

In the following we briefly review the relevant results on group consensus. In [24], the authors addressed the group consensus in continuous-time networks with fixed and undirected topology by using Laplacian spectral theory. Based on some double-tree-form transformations, the result was extended in [25,26] to allow for switching topologies and communication delays. A restriction in the above works that the sum of adjacent weights from each agent in one group to all agents in another group is equal to zero at any time was further relaxed in [23]. For discrete-time multi-agent systems with fixed and switching topologies, algebraic criterions of group consensus were established in [20,21] by discussing the ergodicity of Markov chains and associated transition matrices. In [22], the author considered an L^1 group consensus problem under fixed network by introducing adapted stochastic inputs. We mention that, in different settings, stochastic consensus problems have been well studied; see e.g. [29–31].

In this Letter, we aim to further investigate the group consensus for discrete-time multi-agent systems and extend the L^1 group consensus results obtained in the prior work [22] to switching topologies. Our motivation to address stochastic inputs, which is a major difference from the work [21], mainly arises from the consideration that the imposed inputs are inevitably subject to many uncertainties inherent in complex systems (for example, environmental noises in autonomous mobile swarms, measurement errors due to imperfect communication channels, and parameter uncertainty in climate models, etc.). In some cases, such as Brownian motors [32] and stochastic resonance [33], maintaining a certain amount of noises in an ordered state is even advantageous. On the other hand, the ability of the consensus algorithm to tolerate gross errors in the input data can be better demonstrated via the study of stochastic consensus (such as L^1 and mean square consensus), meaning that the consensus can still be reached when fluctuation exists. Here, we model the underlying network topologies as weighted directed graphs. To realize L^1 group consensus, namely the expectations of the states of agents within the same group converge asymptotically, external stochastic inputs are assumed to have the same expected value for agents within the same group. We show that the time-varying system achieves L^1 group consensus provided that there exists some $L > 0$ such that the union graph across any L-length time interval has group spanning trees.

The rest of the Letter is organized as follows. In Section 2, we present some preliminaries and the problem formulation. The main results are provided in Section 3. Some simulation results are given in Section 4. Finally, the conclusion is made in Section 5.

2. Problem formulation

A weighted directed graph $G = (V, E, A)$ of order n is composed of a vertex set $V = \{v_1, \dots, v_n\}$, an edge set $E \subseteq V \times V$, and a weighted adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, where $a_{ij} \geqslant 0$ is the element of A on the ith row and jth column. Let $[n] = \{1, 2, \dots, n\}$ be the index set of the vertices. We assume that $(v_j, v_i) \in E$ if and only if $a_{ij} > 0$. A directed path of length l in G from vertex v_i to v_j means that there is a sequence of vertices $v_{r_1}, \dots, v_{r_{l+1}}$ such that $(v_{r_k}, v_{r_{k+1}}) \in E$ for $k = 1, \dots, l$, with $v_{r_1} = v_i$ and $v_{r_{l+1}} = v_j$. If there is a vertex v_i such that for all other vertices v_j there is a directed path from v_i to v_j, then graph G is said to have a directed spanning tree with root v_i. For any vertex $v_i \in V$, if $(v_i, v_i) \in E$, it is called a self-loop on vertex v_i.

For a vector $x \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$, denote by $\|x\|$ a vector norm of x and $\|A\|$ the induced matrix norm of A, respectively. A matrix $P = (p_{ij}) \in \mathbb{R}^{n\times n}$ is called a stochastic matrix if $p_{ij} \geqslant 0$ for all i, j, and $\sum_{j=1}^n p_{ij} = 1$ for $i = 1, \dots, n$. Naturally, given a nonnegative (not necessarily stochastic) matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, there is a corresponding weighted directed graph $G = (V, E, A)$ in such a way that A is specified as the weighted adjacency matrix. This graph will be denoted by $G[A]$, meaning that it is induced by A.

The multi-agent system to be studied in this Letter is composed of a network G of n agents, which may be divided into K groups. Specifically, a partition $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_K\}$ of the index set $[n]$ is a sequence of subsets of $[n]$ such that $\bigcup_{k=1}^K \mathcal{S}_k = [n]$ and $\mathcal{S}_k \cap \mathcal{S}_{k'} = \emptyset$ for $k \neq k'$. $\mathcal{S}$ naturally induces a partition of the graph G. For a given partition $\mathcal{S}$, the graph G is said to have group spanning trees [21] with respect to $\mathcal{S}$ if for each group $\mathcal{S}_k$, there is a vertex $v_k \in V$ such that there exist paths in G from v_k to all vertices in $\mathcal{S}_k$. The vertex v_k is called the root of the group $\mathcal{S}_k$. Notice that the group spanning tree is a weaker definition than the spanning tree since the root of a group is not necessarily the same for all groups.

Let $x_i(t) \in \mathbb{R}$ denote the state of the ith agent $v_i \in V$ at time t. Denote $x(t) = (x_1(t), \dots, x_n(t))^T$. The group consensus [24] means that for any initial state $x(0) \in \mathbb{R}^n$, the states of agents satisfy

$$\lim_{t\to\infty} \big|x_i(t) - x_j(t)\big| = 0,$$

for all $i, j \in \mathcal{S}_k$ and $k = 1, \dots, K$. Analogously, we say a system reaches an L^1 group consensus if for any initial state $x(0) \in \mathbb{R}^n$, the states of agents satisfy

$$\lim_{t\to\infty} \mathbb{E}\big|x_i(t) - x_j(t)\big| = 0,$$

for all $i, j \in \mathcal{S}_k$ and $k = 1, \dots, K$.

The discrete-time updating rule of the multi-agent system is described by

$$x_i(t+1) = \sum_{j=1}^n p_{ij}(t) x_j(t) + q_i(t), \quad i \in \mathcal{S}_k, \quad k = 1, \dots, K, \tag{1}$$

where $P(t) = (p_{ij}(t)) \in \mathbb{R}^{n\times n}$ is a stochastic matrix and $G[P(t)]$ corresponds to the communication topology at time t. $q_i(t) \in \mathbb{R}$ is an external stochastic input such that at any time t

$$\mathbb{E}q_i(t) = \mathbb{E}q_j(t), \tag{2}$$

for all $i, j \in \mathcal{S}_k$, $k = 1, \dots, K$.

Group ergodicity coefficient of a stochastic matrix $P \in \mathbb{R}^{n\times n}$ with respect to a partition $\mathcal{S}$ is defined as [34]

$$\mu_{\mathcal{S}}(P) = \min_{1\leqslant k\leqslant K} \min_{i,j\in\mathcal{S}_k} \sum_{k'=1}^n \min\{p_{ik'}, p_{jk'}\}.$$

To analyze the convergence of the multi-agent system (1), we need to estimate some characteristics of infinite product of stochastic matrices by means of group ergodicity coefficient. For this, we need the following lemma whose proof can be carried out following the lines of Theorem 5.1 of [35] or Lemma 1 of [21].

Lemma 1. *For a given partition $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_K\}$, if $P(t) \in \mathbb{R}^{n\times n}$ $(t = 1, 2, \dots)$ is a stochastic matrix and its associated weighted directed graph $G[P(t)]$ has group spanning trees and self-loops on every vertex, then $\mu_{\mathcal{S}}(P(t)P(t-1)\cdots P(1)) > 0$ for $t \geqslant n+1$.*

3. Main results

Given a partition $\mathcal{S}$ as above, a stochastic matrix $P \in \mathbb{R}^{n\times n}$ is said to have inter-group common influence [21] if $\sum_{j\in\mathcal{S}_k} p_{ij}$ is independent of i with $i \in \mathcal{S}_{k'}$ but only depends on the indices k and k'. It is straightforward to check that if two stochastic matrices P and Q have inter-group common influence with respect to the same partition $\mathcal{S}$, the product PQ also has inter-group common influence with respect to $\mathcal{S}$. We assume that the weights $\{p_{ij}(t)\}$ in the multi-agent system (1) satisfy the following assumptions.

Assumption 1. There exists a constant $c > 0$ such that for any i, j, and $t \geqslant 0$, either $p_{ij}(t) = 0$ or $p_{ij}(t) \geqslant c$ holds. Moreover, $p_{ii}(t) \geqslant c$ holds for all i and $t \geqslant 0$.

Assumption 2. $P(t)$ has inter-group common influence for all $t \geqslant 0$.

Under Assumption 1, the weighted directed graph $G[P(t)]$ has self-loops on every vertex. Assumption 2 implies that for any pair of groups $\mathcal{S}_k$ and $\mathcal{S}_{k'}$, either there are no edges from $\mathcal{S}_k$ to $\mathcal{S}_{k'}$; or for each vertex in $\mathcal{S}_{k'}$, there are at least one edge from $\mathcal{S}_k$ to it.

We now define the L^1 group consensus manifold as follows

$$\mathcal{M}_{\mathcal{S}} = \Big\{x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \;\Big|\; \mathbb{E}x_i = \mathbb{E}x_j, \text{ for all } i, j \in \mathcal{S}_k,\\ k = 1, \ldots, K\Big\}.$$

It is clear that the system achieves an L^1 group consensus if and only if $\lim_{t\to\infty} x(t) \in \mathcal{M}_{\mathcal{S}}$.

Proposition 1. *For a given partition $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_K\}$, if Assumption 2 holds, then the L^1 group consensus manifold $\mathcal{M}_{\mathcal{S}}$ is invariant under protocol (1) and (2).*

Proof. By Assumption 2, we define

$$b_{k',k}(t) = \sum_{j\in\mathcal{S}_k} p_{ij}(t),$$

for any $i \in \mathcal{S}_{k'}$. It is easy to see that $B(t) = (b_{k',k}(t)) \in \mathbb{R}^{K\times K}$ is a stochastic matrix for all $t \geqslant 0$. Assume that $x(t) \in \mathcal{M}_{\mathcal{S}}$, we need to show that $x(t+1) \in \mathcal{M}_{\mathcal{S}}$ through (1) and (2).

Let $x_k(t)$ be the identical state of the vertices in group $\mathcal{S}_k$ at time t. In addition, let $q_k(t)$ be the identical input for the vertices in group $\mathcal{S}_k$ at time t. For any $i \in \mathcal{S}_{k'}$, we obtain

$$\begin{aligned}\mathbb{E}x_i(t+1) &= \sum_{k=1}^{K}\sum_{j\in\mathcal{S}_k} p_{ij}(t)\mathbb{E}x_j(t) + \mathbb{E}q_i(t)\\ &= \sum_{k'=1}^{K} b_{k',k}(t)\mathbb{E}x_k(t) + \mathbb{E}q_{k'}(t),\end{aligned}$$

which is identical with respect to any $i \in \mathcal{S}_{k'}$. Thus the proof is completed. □

To solve the L^1 group consensus problem, we need to introduce the group Hajnal diameter and a generalized Hajnal inequality. For a matrix A (not necessarily stochastic) with row vectors $A_1, \ldots, A_n$ and a given partition $\mathcal{S}$, the group Hajnal diameter is defined as [34]

$$\mathrm{diam}_{\mathcal{S}}(A) = \max_{1\leqslant k\leqslant K}\max_{i,j\in\mathcal{S}_k} \|A_i - A_j\|,$$

for some vector norm $\|\cdot\|$. The following inequalities extend a classical Hajnal's theorem. We omit the proof since it can be read out from [21] essentially.

Lemma 2. *Suppose $x \in \mathbb{R}^n$ is a column vector, $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ are two stochastic matrices. For a given partition $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_K\}$, if both P and Q have inter-group common influence, then*

$$\mathrm{diam}_{\mathcal{S}}(PQ) \leqslant \big(1 - \mu_{\mathcal{S}}(P)\big)\cdot \mathrm{diam}_{\mathcal{S}}(Q),$$

and

$$\mathrm{diam}_{\mathcal{S}}(Px) \leqslant \big(1 - \mu_{\mathcal{S}}(P)\big)\cdot \mathrm{diam}_{\mathcal{S}}(x).$$

Let $Q(t) = (q_1(t), \ldots, q_n(t))^T$. The system (1) can be rewritten as

$$x(t+1) = P(t)x(t) + Q(t). \tag{3}$$

Theorem 1. *Suppose that Assumptions 1 and 2 hold. For a given partition $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_K\}$, if there exists an integer $L > 0$ such that for any L-length time interval $[t, t+L)$, the union graph $G[\sum_{l=t}^{t+L-1} P(l)]$ has group spanning trees, then the system (3) with (2) achieves an L^1 group consensus as long as for any i the sequence $\{x_i(t){:}\ t \in \mathbb{N}\}$ does not change sign.*

Proof. From (3), we obtain

$$x(t+1) = P(t)P(t-1)\cdots P(0)x(0) + \left(\sum_{l=0}^{t-1} P(t)P(t-1)\cdots P(l+1)Q(l)\right) + Q(t). \tag{4}$$

It follows from Assumption 1 that all diagonal elements of $P(t)$ are positive for $t \geqslant 0$. Therefore, every edge that appears in the union graph $G[\sum_{l=t}^{t+L-1} P(l)]$ will also appear in the graph $G[P(t+L-1)P(t+L-2)\cdots P(l)]$, which hence has group spanning trees as well as positive diagonal elements for any $t \geqslant 0$.

By Lemma 1, there exists an integer $N > 0$ such that for all $t \geqslant 0$,

$$\mu_{\mathcal{S}}\big(P(t+NL-1)P(t+NL-2)\cdots P(t)\big) > 0.$$

Further, from Assumption 1, there exists some $0 < \eta < 1$ such that for any $t \geqslant 0$,

$$\mu_{\mathcal{S}}\big(P(t+NL-1)P(t+NL-2)\cdots P(t)\big) \geqslant \eta \tag{5}$$

holds.

On the other hand, we have

$$\mathrm{diam}_{\mathcal{S}}\big(\mathbb{E}Q(l)\big) = 0 \tag{6}$$

for all $0 \leqslant l \leqslant t$ in view of (2). By using Assumption 2 and the discussion in the beginning of this section, one can see that the product $P(t)P(t-1)\cdots P(l+1)$ has inter-group common influence for all $0 \leqslant l \leqslant t-1$. Therefore, Lemma 2 and (6) yield that

$$\begin{aligned}&\mathrm{diam}_{\mathcal{S}}\big(P(t)P(t-1)\cdots P(l+1)\mathbb{E}Q(l)\big)\\ &\quad\leqslant \big(1 - \mu_{\mathcal{S}}\big(P(t)P(t-1)\cdots P(l+1)\big)\big)\cdot \mathrm{diam}_{\mathcal{S}}\big(\mathbb{E}Q(l)\big) = 0,\end{aligned} \tag{7}$$

for all $0 \leqslant l \leqslant t-1$.

Combining (4), (6) and (7), we have

$$\mathrm{diam}_{\mathcal{S}}\big(\mathbb{E}x(t+1)\big) \leqslant \mathrm{diam}_{\mathcal{S}}\big(P(t)P(t-1)\cdots P(0)\mathbb{E}x(0)\big).$$

For any t, we can represent $t = kNL + l_0$ for some $0 \leqslant l_0 < k$. If $l_0 = 0$, we obtain

$$\mathrm{diam}_{\mathcal{S}}\big(\mathbb{E}x(t+1)\big) \leqslant (1-\eta)^k \cdot \mathrm{diam}_{\mathcal{S}}\big(\mathbb{E}x(0)\big) \to 0,$$

as $k \to \infty$ (i.e. $t \to \infty$), by employing Lemma 2 and (5). If $l_0 \geqslant 1$, we can derive similarly

$$\begin{aligned}&\mathrm{diam}_{\mathcal{S}}\big(\mathbb{E}x(t+1)\big)\\ &\quad\leqslant (1-\eta)^k \cdot \mathrm{diam}_{\mathcal{S}}\big(P(l_0-1)P(l_0-2)\cdots P(0)\mathbb{E}x(0)\big) \to 0,\end{aligned}$$

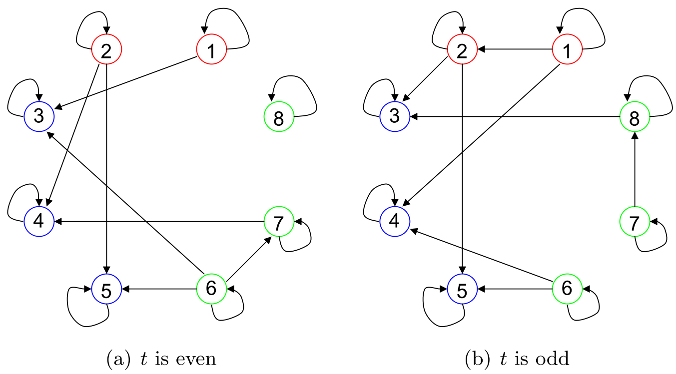

Fig. 1. A network of eight agents with topologies switching in (a) and (b). Three groups $\{1, 2\}$, $\{3, 4, 5\}$ and $\{6, 7, 8\}$ are coded by three different colors for illustration. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

as $t \to \infty$. In view of the sign condition we conclude that the proof of Theorem 1 is completed. □

By the above theorem, we can obtain a group consensus in the sense of almost sure convergence.

Corollary 1. *Suppose that Assumptions 1 and 2 hold. For a given partition $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_K\}$, if there exists an integer $L > 0$ such that for any L-length time interval $[t, t + L)$, the union graph $G[\sum_{l=t}^{t+L-1} P(l)]$ has group spanning trees, then there exists a subsequence $\{t_k\}$ of time such that the system (3) with (2) achieves a group consensus almost surely as $t_k \to \infty$.*

4. Simulation study

In this section, we present numerical simulations to illustrate the effectiveness of the proposed theoretical results. We consider a multi-agent system with eight agents $\{1, 2, \ldots, 8\}$ and its graph topology is switching among the topologies given in Fig. 1(a) and Fig. 1(b) periodically. The associated weighted adjacency matrices corresponding to Fig. 1(a) and Fig. 1(b) are taken as

$$P(t\ \text{even}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$P(t\ \text{odd}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 1/3 \\ 1/3 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix},$$

respectively.

The above system has a partition $\mathcal{S}$ with $\mathcal{S}_1 = \{1, 2\}$, $\mathcal{S}_2 = \{3, 4, 5\}$ and $\mathcal{S}_3 = \{6, 7, 8\}$. Note that neither of the two graphs in Fig. 1(a) and Fig. 1(b) has group spanning trees. However, the union graph of them has group spanning trees and the roots of groups $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are 1, 2 and 6, respectively. It is easy to check that Assumptions 1 and 2 hold. We take initial state as $x(0) = (1, 4, 7, 2, 6, 3, 5, 8)^T$. In what follows, we instantiate the general inputs $Q(t)$ of system (3) in two examples.

Example 1. For any $t \geqslant 0$, we take

$$q_1(t), q_2(t) = \begin{cases} 1, & \text{with probability } 0.5, \\ -1, & \text{with probability } 0.5, \end{cases}$$

$$q_3(t), q_4(t), q_5(t) = \begin{cases} 2, & \text{with probability } 0.5, \\ -2, & \text{with probability } 0.5, \end{cases}$$

$$q_6(t), q_7(t) \sim \text{Poi}\left(\frac{1}{t^2+1}\right), \quad \text{and} \quad q_8(t) \sim \text{Exp}\left(t^2+1\right),$$

where all the random variables involved are independent, $\text{Poi}(\cdot)$ and $\text{Exp}(\cdot)$ refer to Poisson and exponential distributions, respectively. We have $\mathbb{E}q_i(t) = 0$ for $i = 1, \ldots, 5$, $\mathbb{E}q_6(t) = \mathbb{E}q_7(t) = \mathbb{E}q_8(t) = 1/(t^2+1)$. Therefore, all assumptions in Theorem 1 are satisfied. The state trajectories of the agents are shown in Fig. 2(a), and one can observe that L^1 group consensus is achieved asymptotically. In Fig. 2(b) the dynamical behavior of $\text{diam}_{\mathcal{S}}(x(t))$ is plotted. The convergence of $\text{diam}_{\mathcal{S}}(x(t))$ to zero verifies the L^1 group consensus.

Example 2. We take the inputs as autoregressive processes: AR(1) processes. Specifically, for $t \geqslant 1$ and $i = 1, 2$

$$q_i(t) = \frac{1}{2} q_i(t-1) + \varepsilon(t),$$

for $i = 3, 4, 5$,

$$q_i(t) = \frac{1}{3} q_i(t-1) + \varepsilon(t),$$

and for $i = 6, 7, 8$,

$$q_i(t) = \frac{1}{4} q_i(t-1) + \varepsilon(t),$$

where $\varepsilon(t) \sim \text{N}(0, 1)$ is white noise. For all i, $q_i(0) = 0$. It is known that these processes are stationary and $\mathbb{E}q_i(t) = 0$ for all i. Therefore, all assumptions in Theorem 1 are satisfied. The state trajectories of the agents are shown in Fig. 3(a), and the behavior of $\text{diam}_{\mathcal{S}}(x(t))$ is plotted in Fig. 3(b). Both of them indicate that the L^1 group consensus is achieved asymptotically.

5. Concluding remarks

In this Letter, we have provided a theoretical analysis for a discrete-time multi-agent system under switching topologies and external stochastic inputs. L^1 group consensus criteria are deduced for the multi-agent system by using ergodicity theory and non-negative matrix analysis. The intriguing physical insight behind is that intra-group consensus in a system of autonomous particles can be realized in the sense of L^1 convergence even when the network topology switches and stochastic fluctuation exists. Numerical simulations are provided to illustrate the availability of the obtained results. It is essential to find a suitable partition of the system in question, which satisfies the inter-group common influence and group spanning trees properties. A number of network partition/community detection techniques have been developed in statistical physics, which lead to fast and effective algorithms with application to real network data sets (see e.g. [36–39]). Hence, it is viable to check our consensus criteria and apply them to analyze complex systems in the real world. Future work may focus on deeper mathematical study of the algorithm, in order to relax the inter-group common influence condition or to design further improvement such as finite-time group consensus algorithms, etc.

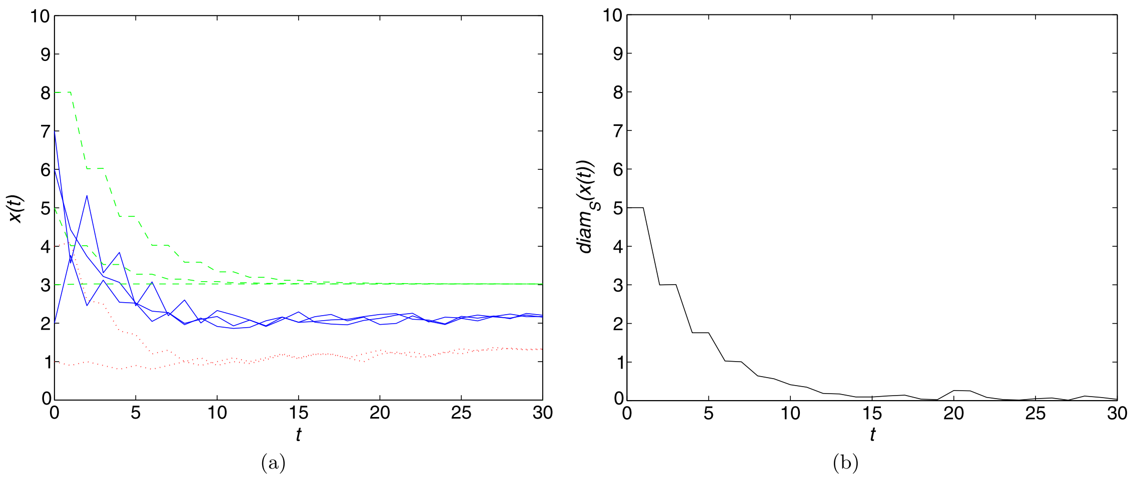

Fig. 2. (a) State trajectories of the agents 1, 2 (red dotted lines), agents 3, 4, 5 (blue solid lines), and agents 6, 7, 8 (green dashed lines). (b) Behavior of $\mathrm{diam}_{\mathcal{S}}(x(t))$ for Example 1. The plots of simulation results correspond to the average of 30 independent simulation runs.

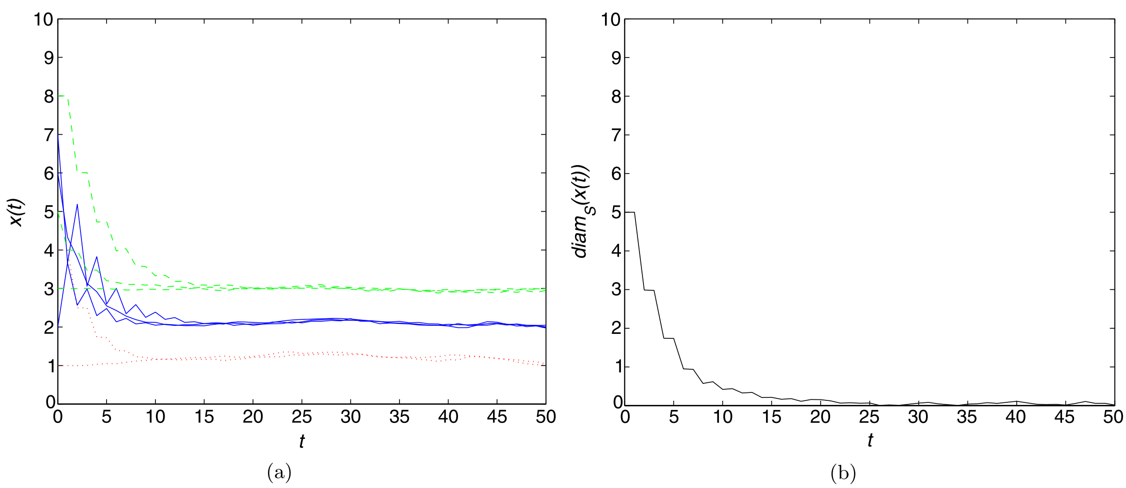

Fig. 3. (a) State trajectories of the agents 1, 2 (red dotted lines), agents 3, 4, 5 (blue solid lines), and agents 6, 7, 8 (green dashed lines). (b) Behavior of $\mathrm{diam}_{\mathcal{S}}(x(t))$ for Example 2. The plots of simulation results correspond to the average of 30 independent simulation runs.

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