

Optimal Policies for Inventory Systems with Concave Ordering Costs

Yimin Yu

College of Business

City University of Hong Kong, Kowloon, Hong Kong

yiminyu@cityu.edu.hk

Saif Benjaafar

Industrial and Systems Engineering

University of Minnesota, Minneapolis, MN 55455

saif@umn.edu

Abstract

In this paper we characterize the structure of optimal policies for periodic review inventory systems with concave ordering costs. By extending the Scarf (1958) model to systems with piecewise linear concave ordering costs, we show the optimality of generalized (s, S) policies for general demand distributions. We do so by (a) introducing a conditional monotonicity property for the order-up-to levels, (b) transforming the problem into one with n dimensions, and (c) by applying the notion of \mathbf{K} -convexity in \mathbb{R}^n to this problem. We show how our results can be extended to systems with fixed leadtimes or lost sales.

Key words: inventory/production systems, dynamic programming, K -convexity, generalized (s, S) policy

1 Introduction

Most of the literature on inventory systems usually assumes a linear ordering cost or a linear ordering cost with a setup cost. As Scarf (1963) argues, “This type of cost functions has appeared in inventory theory not necessarily because of its realism, but because it provides one of the few examples of cost functions with a decreasing average cost for which the analysis of inventory policies is relatively easy.” In this paper, we consider inventory systems with general concave ordering cost functions, where the type of ordering costs described in Scarf (1985) is a special case under our setting. The class of concave ordering cost functions is a special type of a decreasing average cost and there are many examples of concave ordering costs in practice. Consider the following examples.

Quantity Discounts: Quantity discounts provide a practical foundation for coordinating inventory decisions in supply chains. Sellers usually employ quantity discount schemes or contracts to give buyers the incentive to buy more. That is, the larger the order is, the lower the marginal price will be. Ordering costs with quantity discounts can usually be expressed by piecewise linear concave functions: first, there is a setup cost; then the first few items have the same per-unit cost; the next few items have a lower per-unit cost, and so on.

The Effect of Economies of Scale: In economics, one of the common assumptions on production functions is that they have the economies of scale feature. Basically, the more a firm produces the same item, the more efficient the production technology will be. The transportation costs in supply chains also exhibit economies of scale: the more volume of goods to be shipped, the cheaper the marginal cost will be. Concave functions are an important class of functions exhibiting the feature of economies of scale.

Procurement with Multiple Suppliers: In many cases in practice, there are multiple suppliers available for a buyer in practice. Different suppliers differ in their cost structures. Usually local suppliers offer relatively lower setup costs but with higher per-unit costs and overseas or distant suppliers offer higher setup costs but with lower per-unit costs. Hence, if the buyer chooses the suppliers optimally, the resulting ordering cost is a piecewise linear concave function. A related case of a buyer that purchases from both long-term suppliers and spot markets is treated in Yi and

Scheller-Wolf (2003) and the references therein. There are many other examples of concave costs due to the availability of multiple choices of labor and production, see Fox *et al* (2006) for examples and references therein.

In his seminal paper, Scarf (1958) proves that the (s, S) policy is optimal for an inventory system with a fixed ordering cost and a unit ordering cost and does it by introducing the notion of K -convexity. Note that this type of ordering cost is a special case of concave ordering costs. Karlin (1958) analyzes the optimal ordering policy for a one-period inventory problem with concave ordering costs. Scarf (1963) points out that it is difficult to generalize the result to the dynamic multiperiod setting. There has been only limited research on inventory systems with concave ordering costs. Porteus (1971) analyzes inventory systems with piecewise linear concave ordering costs. He shows that a generalized (s, S) policy is optimal for a multi-period periodic review inventory system under some mild assumption on cost functions and that demand has a one-sided Polya density. He does it by introducing a generalized notion of K -convexity called quasi- K -convexity. However, the class of one-sided Polya densities does not include many densities encountered in practice, for example, the normal distribution, beta distribution and most gamma distributions, although it does include the exponential distribution and all its finite convolutions. Porteus (1972) also shows that the generalized (s, S) policy is optimal for uniform demand distributions.

Fox *et al.* (2006) consider the optimal policy for an inventory system with two suppliers: the buyer incurs a high variable cost but negligible fixed cost for the first supplier (HVC) and a lower variable cost and but a substantial fixed cost for the second supplier (LVC). The resulting ordering cost is a two-piece linear concave function. They show that the optimal policy is a (s, S_{HVC}, S_{LVC}) policy, which is a special case of the generalized (s, S) policy, under the condition that the demand density is log-concave. Their proof relies on K -convexity and quasi-convex properties since they consider a two-piece linear concave function. Although the class of log-concave densities is less restrictive than the class of one-sided Polya densities, it still only covers a limited range of distributions. Furthermore, their results do not cover general piecewise linear concave ordering costs and it is hard to generalize their approach to cases with positive leadtimes as they point out. Hence, whether or not the generalized (s, S) policy is optimal for general demand distributions remained an open question.

Recently, Chen et al (2009) consider joint pricing and inventory control for inventory systems with concave ordering costs. They utilize quasi-K-convexity to show that the optimal policy is a generalized (s, S, p) policy when demand distributions are Polya or uniform. Another related paper is Yi and Scheller-Wolf (2003), where they also consider a two-supplier inventory problem: the buyer has a long-term contract from a regular supplier with a minimum and maximum purchasing quantity, and the buyer can also purchase from a spot market that has no quantity limitation but with a fixed entry fee. They show that the structure of the optimal policy is similar to (s, S) policy and can be described by four pieces of information: s^* , s^b , $S^a(x)$ and S^b such that it is optimal to compare inventory level with s^* , s^b , and order up to $S^a(x)$ from the regular supplier and/or S^b from the spot market. Their proof relies on a closure property of K -convexity. Note that the ordering cost in their case is no longer concave since they assume a limited capacity for the regular supplier and the corresponding optimal policy is not a generalized (s, S) policy.

In contrast to the existing literature, in order to show the structure of optimal policies for inventory systems with concave ordering costs, we show a conditional monotone property on the order-up-to levels. We then introduce a monotone condition that ensures the optimality of a generalized (s, S) policy. In order to show that this monotone condition indeed holds, we use the notion of \mathbf{K} -convexity in \mathbb{R}^n . To do so, we first transform our problem into an n -dimensional problem (this n -dimensional problem is equivalent to an appropriately defined inventory problem with n suppliers). We then show that the optimal value function for this problem is \mathbf{K} -convex in \mathbb{R}^n , which in turn implies that the generalized (s, S) policy is optimal. In contrast to previous papers, our results hold for any demand distribution function under our setting. Also, our results can be readily extended to systems with fixed leadtimes or lost sales.

Our paper is related to the literature on \mathbf{K} -convexity in \mathbb{R}^n . This literature focuses on utilizing \mathbf{K} -convexity in \mathbb{R}^n for multiple products with setup costs. However, since generally \mathbf{K} -convexity in \mathbb{R}^n cannot be preserved under minimization, we need restrictive conditions in order to preserve \mathbf{K} -convexity in \mathbb{R}^n . For these applications of \mathbf{K} -convexity in \mathbb{R}^n , see John (1967), Liu and Esogbue (1999), Ohno and Ishigaki (2001), and the detailed review in Gallego and Sethi (2005).

2 Inventory Systems with Concave Ordering Costs

We consider a single product single stage inventory problem with multiple periods, stochastic demands, and zero leadtime. The assumption on zero leadtime is not critical and is made for ease of exposition (see Section 4 for extensions). Demand ξ_t in each period t is a continuous random variable with $\mathbb{E}[\xi_t] < \infty$ and distribution function $F_t(x), x \geq 0$, where $t = 1, \dots, T$ and T corresponds to the length of the planning horizon. Demands in different periods are independent but not necessarily identically distributed (i.e., demand can be time-varying). Inventory is replenished from an outside supplier immediately (i.e., with zero leadtime) with ample stock. Demand is satisfied from on-hand inventory, if any is available; otherwise it is backordered. In each period, the inventory manager must decide on the quantity to order to minimize the expected discounted cost over the entire planning horizon. There are three types of costs in each period t : (1) an ordering cost $c(z)$ if we order $z, z \geq 0$ quantity, (2) a holding cost $h_t(x^+)$ and (3) a backordering cost $b_t(x^-)$ given the inventory level x in period t , where $x^+ = \max\{0, x\}$ and $x^- = \max\{0, -x\}$. Finally, we allow a discount factor $\alpha \in (0, 1]$.

In order to simplify our presentation, we first consider a piecewise linear concave ordering cost $c(\cdot)$ with n linear pieces. Specifically, we can express

$$c(x) = \min_{i=1, \dots, n} \{K_i \delta(x) + c_i x\},$$

with $0 \leq K_1 < K_2 < \dots < K_n$ and $c_1 > c_2 > \dots > c_n \geq 0$, where δ is defined as follows

$$\delta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases} \quad (1)$$

We will discuss how we can deal with general concave ordering costs at the end of this section and time-varying costs in the section on extensions (see Section 4).

Let

$$g_t(x - \xi_t) = \begin{cases} h_t(x - \xi_t) & \text{if } x \geq \xi_t, \\ b_t(\xi_t - x) & \text{otherwise.} \end{cases}$$

Let x_t be the starting inventory level and y_t be the post-ordering inventory level for period t . It is clear that $x_{t+1} = y_t - \xi_t$. Given $x_1, \dots, x_T, y_1, \dots, y_T$, i.e., the ordering quantities are $q_t = y_t - x_t, t = 1, \dots, T$, the expected discounted total cost is given by

$$\mathbb{E} \left\{ \sum_{t=1}^T \alpha^t [c(y_t - x_t) + g_t(y_t - \xi_t)] \right\}. \quad (2)$$

Let $v_t^*(x)$ be the value function (the optimal expected discounted cost) in period t when the inventory level in period t is x . Then the corresponding dynamic programming formulation is given by

$$v_t^*(x) = \min_{y \geq x} \{c(y - x) + \mathbb{E}g_t(y - \xi_t) + \alpha \mathbb{E}v_{t+1}^*(y - \xi_t)\}. \quad (3)$$

Finally we let $v_{T+1}^*(x) = 0$ for all x .

Assumption 1 *We assume that $L_t(y) = \mathbb{E}g_t(y - \xi_t)$ is convex in y and finite for any y .*

For example, this assumption is satisfied if h_t and b_t are linear and $\mathbb{E}[\xi_t] < \infty$. The finiteness of L_t ensures that L_t is continuous on $(-\infty, \infty)$ (by the dominated convergence theorem).

Let

$$H_t(y) = \mathbb{E}g_t(y - \xi_t) + \alpha \mathbb{E}v_{t+1}^*(y - \xi_t).$$

Then the optimality equation is given by

$$v_t^*(x) = \min_{y \geq x} [c(y - x) + H_t(y)]. \quad (4)$$

Given x , let $y_t(x)$ be the smallest minimizer of $c(y - x) + H_t(y)$, i.e.,

$$y_t(x) = \min \arg \min_{y \geq x} \{c(y - x) + H_t(y)\}. \quad (5)$$

Hence, given the current inventory level is x , it is optimal to order $y_t(x) - x$ quantity in period t .

3 An Equivalent Transformation

In this section we transform the original one dimensional dynamic programming problem into an equivalent n -dimensional dynamic programming problem. This transformation plays a key role in characterizing the structure of the optimal policy. The transformation can be done as follows.

Note that $c(x) = \min_{i=1, \dots, n} \{K_i \delta(x) + c_i x\}$, with $K_1 < K_2 < \dots < K_n$ and $c_1 > c_2 > \dots > c_n$. We can think of there being n suppliers and each supplier i having fixed ordering cost K_i and variable cost c_i . We can also think of determining the optimal quantity in each period, as deciding on the quantities $q_{i,t}$ that we need to order from each supplier i for $i = 1, \dots, n$ to minimize the expected discounted cost.

In order to transform our problem into the multiple sourcing problem with n suppliers, we apply the following transformation. We define the variables $x_{i,t}, i = 1, \dots, n$ such that $\sum_{i=1}^n x_{i,t} = x_t$. Let $i^*(q_t) = \min \arg \min_{i=1, \dots, n} \{K_i \delta(q_t) + c_i q_t\}$. Then we can define

$$y_{i,t} = \begin{cases} x_{i,t} + q_t, & \text{for } i = i^*(q_t), \\ x_{i,t}, & \text{otherwise} \end{cases} \quad (6)$$

i.e., we have $y_{i^*(q_t),t} - x_{i^*(q_t),t} = q_t$ and $y_{i,t} = x_{i,t}$ for all $i \neq i^*(q_t)$. As a result, we also have $\sum_{i=1}^n y_{i,t} = y_t$. In this way, we can ensure that

$$\sum_{i=1}^n K_i \delta(y_{i,t} - x_{i,t}) + \sum_{i=1}^n c_i (y_{i,t} - x_{i,t}) = c(y_t - x_t) = c(q_t).$$

Then we can reformulate the expected discounted cost in (2) as

$$\mathbb{E} \left\{ \sum_{t=1}^T \alpha^t \left[\sum_{i=1}^n K_i \delta(y_{i,t} - x_{i,t}) + \sum_{i=1}^n c_i (y_{i,t} - x_{i,t}) + g_t \left(\sum_{i=1}^n y_{i,t} - \xi_t \right) \right] \right\}. \quad (7)$$

Since we can choose arbitrary $x_{i,t}$ as long as $\sum_{i=1}^n x_{i,t} = x_t$, given $\mathbf{y}_t = (y_{1,t}, \dots, y_{n,t})$ and demand ξ_t we can assign the starting state variable in period $t+1$ for supplier i as $x_{i,t+1} = r_i(\mathbf{y}_t, \xi_t)$. Note that r_i can be arbitrary functions as long as $\sum_{i=1}^n r_i(\mathbf{y}_t, \xi_t) = \sum_{i=1}^n y_{i,t} - \xi_t$. Then we can transform the expected cost in (7) as follows (we do it by assigning $-\alpha^{t+1} \sum_{i=1}^n c_i x_{i,t+1}$ to period

t and using the fact that $x_{i,t+1} = r_i(\mathbf{y}_t, \xi_t)$:

$$\mathbb{E} \left\{ \sum_{t=1}^T \alpha^t \left[\sum_{i=1}^n K_i \delta(y_{i,t} - x_{i,t}) + \sum_{i=1}^n c_i (y_{i,t} - x_{i,t}) + g_t \left(\sum_{i=1}^n y_{i,t} - \xi_t \right) \right] \right\} \quad (8)$$

$$= \sum_{t=1}^T \alpha^t \left[\sum_{i=1}^n K_i \delta(y_{i,t} - x_{i,t}) + W_t(\mathbf{y}_t) \right] - \alpha \sum_{i=1}^n c_i x_{i,1}, \quad (9)$$

where $W_t(\mathbf{y}_t) = \mathbb{E}[\sum_{i=1}^n c_i y_{i,t} + g_t(\sum_{i=1}^n y_{i,t} - \xi_t) - \alpha \sum_{i=1}^n c_i r_i(\mathbf{y}_t, \xi_t)]$. Since we have $x_{i,t+1} = r_i(\mathbf{y}_t, \xi_t)$, $W_t(\mathbf{y}_t) = \mathbb{E}[\sum_{i=1}^n c_i y_{i,t} + g_t(\sum_{i=1}^n y_{i,t} - \xi_t) - \alpha \sum_{i=1}^n c_i x_{i,t+1}]$. A similar transformation technique is used in Veinott (1965).

Let $K(\mathbf{y} - \mathbf{x}) = \sum_{i=1}^n K_i \delta(y_i - x_i)$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $K_i \geq 0$ for $i = 1, \dots, n$. Given the starting inventory level $\mathbf{x} = (x_1, \dots, x_n)$ for the n suppliers in period t , then the corresponding dynamic programming formulation under the transformed state to minimize the cost in (7) is given by

$$\tilde{v}_t^*(\mathbf{x}) = \min_{\mathbf{y} \geq \mathbf{x}} \{K(\mathbf{y} - \mathbf{x}) + W_t(\mathbf{y}) + \alpha \mathbb{E}[\tilde{v}_{t+1}^*(\mathbf{r}(\mathbf{y}, \xi_t))]\}, \quad (10)$$

where $\mathbf{r}(\mathbf{y}, \xi_t) = (r_1(\mathbf{y}, \xi_t), \dots, r_n(\mathbf{y}, \xi_t))$. We can reformulate this dynamic program as follows

$$\tilde{v}_t^*(\mathbf{x}) = \min_{\mathbf{y} \geq \mathbf{x}} \{K(\mathbf{y} - \mathbf{x}) + G_t(\mathbf{y})\}, \quad (11)$$

where $G_t(\mathbf{y}) = W_t(\mathbf{y}) + \alpha \mathbb{E}[\tilde{v}_{t+1}^*(\mathbf{r}(\mathbf{y}, \xi_t))]$. By the equivalent transformation of the expected cost, it follows that $v_t^*(x) = \tilde{v}_t^*(\mathbf{x}) - \alpha \sum_{i=1}^n c_i x_{i,1}$ with the understanding that $x = \sum_{i=1}^n x_i$ for each period t . But since $\alpha \sum_{i=1}^n c_i x_{i,1}$ is a constant, the optimal inventory policy for formulation (10) is also optimal for formulation (3). Let

$$\mathbf{y}_t(\mathbf{x}) = \min \arg \min_{\mathbf{y} \geq \mathbf{x}} \{K(\mathbf{y} - \mathbf{x}) + G_t(\mathbf{y})\}.$$

We have the following lemma concerning the dynamic programming formulation (10).

Lemma 1 *Given \mathbf{x} , it is optimal to order amount $q_t^* = y_t(\sum_{i=1}^n x_i) - \sum_{i=1}^n x_i$ from supplier $i^*(q_t^*)$ such that $i^*(q_t^*) = \min \arg \min_{\{i=1, \dots, n\}} \{K_i \delta(q_t^*) + c_i q_t^*\}$ and 0 for all other suppliers. Furthermore, the value function $\tilde{v}_t^*(\mathbf{x})$ is independent of any particular choice of \mathbf{r} and hence $\tilde{v}_t^*(\hat{\mathbf{x}}) = \tilde{v}_t^*(\mathbf{x})$ as*

long as $\sum_{i=1}^n x_i = \sum_{i=1}^n \hat{x}_i$.

The proof follows from the fact that $v_t^*(x) = \tilde{v}_t^*(\mathbf{x}) - \alpha \sum_{i=1}^n c_i x_{i,1}$ with the understanding that $\sum_{i=1}^n x_i = x$ for each period t . The fact that $\tilde{v}_t^*(\mathbf{x})$ is independent of any particular choice of \mathbf{r} is because that the expected cost in (9) remains unchanged for different \mathbf{r} .

For simplicity, in the sequel we assume that $\mathbf{r}(\mathbf{x}, \xi) = (\sum_{i=1}^n x_i - \xi, 0, \dots, 0)$, i.e., we use the first coordinate to represent the inventory level. Then under our assumptions, it follows that W_t is convex and hence continuous.

4 The Optimality of Generalized (s, S) Policies

First, we show a conditional monotone property for $y_t(x)$ for any c that is concave.

Theorem 1 *Suppose that $y_t(x) > x$, then $y_t(z) \leq y_t(x)$ for $z \in (x, y_t(x))$.*

Proof. Suppose that for some x , we have $y_t(x) - x > 0$. Let $y_t(x) > z > x$. Since we know that $c(x)$ is concave in x , for $\omega > 0$ we have

$$c(y_t(x) + \omega - z) - c(y_t(x) - z) \geq c(y_t(x) + \omega - x) - c(y_t(x) - x),$$

which implies that

$$\begin{aligned} & c(y_t(x) + \omega - z) + H_t(y_t(x) + \omega) - [c(y_t(x) - z) + H_t(y_t(x))] \\ & \geq c(y_t(x) + \omega - x) + H_t(y_t(x) + \omega) - [c(y_t(x) - x) + H_t(y_t(x))] \geq 0. \end{aligned} \quad (12)$$

Since inequality (12) is true for all $\omega > 0$ and c is continuous at $(0, \infty)$, it follows that $y_t(z) \leq y_t(x)$. ■

As far as we know, this is a new result in the literature. The interesting aspect of this result is that the conditional monotone property in period t holds for general concave ordering costs. However we do not know what will happen for $z \notin (x, y_t(x))$. Next, we describe a monotone condition that is the key to characterizing the structure of optimal policies.

Condition 1 $y_t(x_2) > x_2$ implies that $y_t(x_1) > x_1$ for any $x_1 < x_2$. In words, if it is optimal to order a positive amount when the starting inventory level is x_2 , then it must be optimal to order a positive amount when the starting inventory level is less than x_2 in period t .

It turns out that if we know that $y_t(x_2) > x_2$ implies that $y_t(x_1) > x_1$ for any $x_1 < x_2$, then coupled with Theorem 1, we can show that $y_t(x_1) \geq y_t(x_2)$ for all $x_1 < x_2$ such that $y_t(x_2) > x_2$.

Lemma 2 Suppose that we have Condition 1. Then we must have

- (1) $y_t(x_1) \geq y_t(x_2)$ for all $y_t(x_2) > x_2$ and $x_1 < x_2$.
- (2) There exists some x_0 such that $y_t(x) = x$ for all $x \geq x_0$ and $y_t(x)$ is non-increasing in x for $x \in (-\infty, x_0]$.

Proof. We prove the first part by contradiction. Suppose we have $y_t(x_1) < y_t(x_2)$ given that $y_t(x_2) > x_2$, $y_t(x_1) > x_1$ and $x_1 < x_2$. We differentiate two cases. (1) $y_t(x_1) \leq x_2$. This case is impossible, since under Condition 1, we must have $y_t(y_t(x_1)) > y_t(x_1)$, i.e., $y_t(x_1)$ is not an optimal order-up-to level for x_1 , which violates the optimality of $y_t(\cdot)$. (2) $y_t(x_1) \in (x_2, y_t(x_2))$. This case is impossible since it violates Theorem 1. We know that by Theorem 1, we must have $y_t(x_2) < y_t(x_1)$ since $x_2 \in (x_1, y_t(x_1))$.

Since we have $\lim_{x \rightarrow \infty} H_t(x) = \infty$, it follows that for sufficiently large x , we must have $y_t(x) = x$. Let x_0 the smallest value such that $y_t(x) = x$. Since x_0 is the minimum of such value, then $y_t(x) > x$ for all $x < x_0$ by Condition 1. It follows that $y_t(x)$ is non-increasing in x in that domain by part (1) of this Lemma. It can also be shown that $y_t(x) = x$ for all $x > x_0$, otherwise if $y_t(x) > x > x_0$ then we must have $y_t(x_0) > x_0$ but that contradicts with the definition of x_0 . ■

Theorem 2 If Condition 1 is satisfied, then the optimal inventory policy in period t is a generalized (s, S) policy, i.e., there exists $(s_{m,t}, \dots, s_{1,t}, S_{1,t}, \dots, S_{m,t})$ with $s_{m,t} < s_{m-1,t} < \dots < s_{1,t} \leq S_{1,t} < S_{2,t} < \dots < S_{m,t}$ for some $m \leq n$ such that if $x < s_{m,t}$ then we order up to $S_{m,t}$ and if $x \in [s_{i,t}, s_{i-1,t})$ then we order up to $S_{i-1,t}$ for $i = 2, \dots, m$, and finally we order nothing for $x \geq s_{1,t}$. Hence, we at most have n distinctive such order-up-to levels $S_{i,t}$. Furthermore, if H_t is a continuous function, then it follows that v_t^* is also a continuous function.

Proof. Let $s_{1,t} = \min\{x : H(x) \geq c(y-x) + H_t(y), y > x\}$, i.e., $s_{1,t}$ is the minimum starting inventory level such that it is optimal to order nothing (The existence of $s_{1,t}$ is due to $\lim_{x \rightarrow \infty} H_t(x) = \infty$ and H_t is continuous). It follows that if $x > s_{1,t}$, then it is also optimal to order nothing, otherwise it would violate the definition of $s_{1,t}$. Also if $x < s_{1,t}$, then it must be optimal to order a positive quantity and the post-ordering inventory level must be greater than $s_{1,t}$, otherwise, it would violate Condition 1.

Let

$$\hat{S}_{i,t} = \min \arg \min_{y \geq s_{1,t}} \{H_t(y) + c_i y\},$$

i.e., $\hat{S}_{i,t}$ is the minimum of $H_t(y) + c_i y$ on $[s_{1,t}, \infty)$ (the existence of $\hat{S}_{i,t}$ is due to the continuity of H_t). Since $c_1 > c_2 > \dots > c_n$, it follows that $\hat{S}_{1,t} \leq \hat{S}_{2,t} \leq \dots \leq \hat{S}_{n,t}$ (by the property that $c_i y$ is submodular in (y, i)). For $x \in (-\infty, s_{1,t})$, let

$$v_{i,t}(x) = \min_{y_i \geq x} [c_i y_i + K_i \delta(y_i - x) + H_t(y_i)] - c_i x = \min\{\min_{y_i > x} [c_i y_i + K_i + H_t(y_i)], c_i x + H_t(x)\} - c_i x.$$

We have

$$\min_{y \geq x} \{c(y-x) + H_t(y)\} = \min_{i=1, \dots, n} \{v_{i,t}(x)\}.$$

It follows that for any starting inventory level $x \in (-\infty, s_{1,t})$ (Note that it is optimal to order a positive quantity for such starting inventory level x), it must be optimal to order to one of the levels in $\{\hat{S}_{1,t}, \hat{S}_{2,t}, \dots, \hat{S}_{n-1,t}\}$. Ties can be broken by choosing the smallest solution.

Suppose that for small enough δ it is optimal to order up to $\hat{S}_{i_1,t}$ for some $i_1 \in \{1, \dots, n\}$ for starting inventory level $x \in [s_{1,t} - \delta, s_{1,t})$. If $i_1 = n$, then we are done. Otherwise, let $s_{2,t}$ be the smallest value such that it is optimal to order up to $\hat{S}_{i_1,t}$, i.e., $y_t(x) = \hat{S}_{i_1,t}$ for $x \in [s_{2,t}, s_{1,t})$. We define $S_{1,t} \equiv \hat{S}_{i_1,t}$. Since it is also optimal to order a positive quantity for $x < s_{2,t}$, by Lemma 2, we have $y_t(x) > \hat{S}_{i_1,t}$ for $x < s_{2,t}$. Again, suppose for some small enough δ it is optimal to order-up-to $\hat{S}_{i_2,t}$ for some $i_2 \in \{1, \dots, n\}$ for $x \in [s_{2,t} - \delta, s_{1,t})$. Obviously, we have $i_2 > i_1$ by the conditional monotone property. We define $S_{2,t} \equiv \hat{S}_{i_2,t}$. If $S_{2,t} = \hat{S}_{n,t}$, then we are done. Otherwise by a similar argument, we can iteratively define $s_{i,t}$ and $S_{i,t}$ ($i > 3$) such that it is optimal to order-up-to $S_{i,t}$

for $x \in [s_{i+1,t}, s_i)$ until we have some $s_{m,t}$ and $S_{m,t} = \hat{S}_{n,t}$. Then it follows that if $x < s_{m,t}$, it is optimal to order up to $S_{m,t} = \hat{S}_{n,t}$. It is also clear that $m \leq n$.

The continuity property of v_t^* is due to the inductive assumption that H_t is continuous and the optimality of generalized (s, S) policies. ■

Hence, what remains to be done is to prove that we do have Condition 1 for the optimal policies. Obviously, the K -convexity proposed in Scarf (1958) is not enough, since our ordering costs are piecewise concave functions.

5 **K-Convex Functions in \mathbb{R}^n**

In order to fully characterize the structure of optimal policies for inventory systems with piecewise linear concave ordering costs, we first define a class of **K**-convex functions in \mathbb{R}^n (also see Gallego and Sethi (2005)).

We say $\mathbf{x} \geq \mathbf{y}$ ($\mathbf{x} \leq \mathbf{y}$) if $x_i \geq y_i$ ($x_i \leq y_i$) for all $i = 1, \dots, n$. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **K**-convex in \mathbb{R}^n (for $K_n > K_{n-1} > \dots > K_1$) if for each $\theta \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} \leq \mathbf{y}$,

$$f(\theta\mathbf{x} + (1 - \theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1 - \theta)[f(\mathbf{y}) + K(\mathbf{y} - \mathbf{x})].$$

This **K**-convexity in \mathbb{R}^n clearly is a generalization of K -convexity in \mathbb{R} proposed by Scarf (1958) to \mathbb{R}^n . For applications of **K**-convexity in \mathbb{R}^n , see John (1967), Karlin (1980), Ohno and Ishigaki (2001), Gallego and Sethi (2005) and references therein. However, this literature focuses on the case with multiple products. Similar to K -convexity in \mathbb{R} , we have the following properties for **K**-convex functions in \mathbb{R}^n .

Lemma 3 (Gallego and Sethi 2005) *K -convex functions in \mathbb{R}^n have the following properties:*

1. A convex function in \mathbb{R}^n is **0**-convex, where **0** denotes that $K_i = 0$ for all i .
2. If g is **K**-convex in \mathbb{R}^n , then $g(\mathbf{x} + \mathbf{a})$ is also **K**-convex in \mathbb{R}^n for all $\mathbf{a} \in \mathbb{R}^n$.
3. If g is **K**-convex in \mathbb{R}^n , then it is also **L**-convex in \mathbb{R}^n if $L_i \geq K_i$ for all i .

4. If g_i is \mathbf{K}^i -convex in \mathbb{R}^n for $i = 1, 2$, then $a_1g_1 + a_2g_2$ is $a_1\mathbf{K}^1 + a_2\mathbf{K}^2$ -convex in \mathbb{R}^n for all non-negative a_i .
5. If g is \mathbf{K} -convex in \mathbb{R}^n and $f(\mathbf{x}) = \mathbb{E}[g(\mathbf{x} - \boldsymbol{\xi})] < \infty$, where $\boldsymbol{\xi}$ is a random vector in \mathbb{R}^n , then $f(\mathbf{x})$ is also \mathbf{K} -convex in \mathbb{R}^n .

In the following we show that Condition 1 holds once we have the \mathbf{K} -convexity in \mathbb{R}^n for G_t under the transformed formulation.

Theorem 3 *If G_t is \mathbf{K} -convex in \mathbb{R}^n , then Condition 1 holds for the original formulation as specified in (3). As a result, the optimal inventory policy in every period t is a generalized (s, S) policy.*

Proof. We prove by contradiction. Suppose that we know that $\mathbf{y}_t(\mathbf{x}^2) > \mathbf{x}^2$ and some $\mathbf{x}^1 \leq \mathbf{x}^2$ such that $\mathbf{y}_t(\mathbf{x}^1) = \mathbf{x}^1$. Therefore, we must have $G_t(\mathbf{x}^1) \leq K(\mathbf{y}_t(\mathbf{x}^2) - \mathbf{x}^1) + G_t(\mathbf{y}_t(\mathbf{x}^2))$. Let $\mathbf{y}_t(\mathbf{x}^2) - \mathbf{x}^2 = w_i \mathbf{e}_i$, where $w_i > 0$. We choose $\hat{\mathbf{x}}^1 = \mathbf{x}^2 - (\sum_{i=1}^n (x_i^2 - x_i^1)) \mathbf{e}_i$, where \mathbf{e}_i is the i th unit vector in \mathbb{R}^n (with the i th element equal to 1 and 0 everywhere else). It follows that $G_t(\mathbf{x}^1) = G_t(\hat{\mathbf{x}}^1)$ by virtue of Lemma 1. We can choose some $\theta \in [0, 1]$ such that $\theta \hat{\mathbf{x}}^1 + (1 - \theta) \mathbf{y}_t(\mathbf{x}^2) = \mathbf{x}^2$. Note that we have

$$\begin{aligned}
& \theta G_t(\hat{\mathbf{x}}^1) + (1 - \theta)[G_t(\mathbf{y}_t(\mathbf{x}^2)) + K(\mathbf{y}_t(\mathbf{x}^2) - \hat{\mathbf{x}}^1)] \\
&= \theta G_t(\hat{\mathbf{x}}^1) + (1 - \theta)[G_t(\mathbf{y}_t(\mathbf{x}^2)) + K_i] \\
&\geq G_t(\theta \hat{\mathbf{x}}^1 + (1 - \theta) \mathbf{y}_t(\mathbf{x}^2)) = G_t(\mathbf{x}^2)
\end{aligned}$$

by \mathbf{K} -convexity in \mathbb{R}^n . On the other hand, we have

$$\begin{aligned}
& \theta G_t(\hat{\mathbf{x}}^1) + (1 - \theta)[G_t(\mathbf{y}_t(\mathbf{x}^2)) + K(\mathbf{y}_t(\mathbf{x}^2) - \hat{\mathbf{x}}^1)] \\
&\leq \theta[G_t(\mathbf{y}_t(\mathbf{x}^2)) + K(\mathbf{y}_t(\mathbf{x}^2) - \hat{\mathbf{x}}^1)] + (1 - \theta)[G_t(\mathbf{y}_t(\mathbf{x}^2)) + K(\mathbf{y}_t(\mathbf{x}^2) - \hat{\mathbf{x}}^1)] \\
&= G_t(\mathbf{y}_t(\mathbf{x}^2)) + K(\mathbf{y}_t(\mathbf{x}^2) - \hat{\mathbf{x}}^1).
\end{aligned}$$

It follows that

$$\begin{aligned} G_t(\mathbf{x}^2) &\leq G_t(\mathbf{y}_t(\mathbf{x}^2)) + K(\mathbf{y}_t(\mathbf{x}^2) - \tilde{\mathbf{x}}^1) \\ &= G_t(\mathbf{y}_t(\mathbf{x}^2)) + K_i = G_t(\mathbf{y}_t(\mathbf{x}^2)) + K(\mathbf{y}_t(\mathbf{x}^2) - \mathbf{x}^2), \end{aligned}$$

which contradicts the fact that $\mathbf{y}_t(\mathbf{x}^2) > \mathbf{x}^2$.

Since $\mathbf{y}_t(\mathbf{x})$ depends only on $\sum_{i=1}^n x_i$, hence Condition 1 holds.

The optimality of the generalized (s, S) policy follows from the first part of this theorem, Lemma 1 and Theorem 2. ■

In order to show the optimality of generalized (s, S) policies what is left is to show that $\tilde{v}_t^*(\mathbf{x})$ is \mathbf{K} -convex in \mathbb{R}^n . It is easy to verify that $K(\mathbf{x})$ has the following subadditivity property:

Lemma 4 *We have*

$$K(\mathbf{x}) + K(\mathbf{y}) \geq K(\mathbf{x} + \mathbf{y}), \forall \mathbf{x}, \mathbf{y} \geq \mathbf{0}.$$

This subadditivity property is due to the fact that $\delta(x) + \delta(y) \geq \delta(x + y)$ for all $x, y \geq 0$.

Next we show that \tilde{v}_t^* under the transformed formulation is actually \mathbf{K} -convex in \mathbb{R}^n .

Lemma 5 *$\tilde{v}_t^*(\mathbf{x})$ and $G_t(\mathbf{x})$ are \mathbf{K} -convex in \mathbb{R}^n and continuous in \mathbf{x} . As a result the generalized (s, S) policy is optimal.*

Proof. We prove the result by induction. Note that the result holds for period T since $E[g_T(x - \xi)]$ is convex and continuous in x . Therefore $G_T(\mathbf{x})$ is also convex and continuous in \mathbf{x} , and hence is \mathbf{K} -convex in \mathbb{R}^n . Suppose the result holds for period $t+1$, then it is sufficient to show that the result holds for period t . Since we know that \tilde{v}_{t+1}^* is \mathbf{K} -convex in \mathbb{R}^n and continuous by the inductive assumption, we can show that G_t is continuous and $\alpha\mathbf{K}$ -convex in \mathbb{R}^n . Let $\mathbf{x} \leq \mathbf{y}$. For $\theta \in [0, 1]$,

we have

$$\begin{aligned}
& \theta G_t(\mathbf{x}) + (1 - \theta)[G_t(\mathbf{y}) + \alpha K(\mathbf{y} - \mathbf{x})] \\
&= \theta[W_t(\mathbf{x}) + \alpha \mathbb{E}[\tilde{v}_{t+1}^*(\mathbf{r}(\mathbf{x}, \xi_t))]] + (1 - \theta)[W_t(\mathbf{y}) + \alpha \mathbb{E}[\tilde{v}_{t+1}^*(\mathbf{r}(\mathbf{y}, \xi_t))] + \alpha K(\mathbf{y} - \mathbf{x})] \\
&\geq W_t(\theta \mathbf{x} + (1 - \theta)\mathbf{y}) + \theta \alpha \mathbb{E}[\tilde{v}_{t+1}^*(\mathbf{r}(\mathbf{x}, \xi_t))] + (1 - \theta) \alpha \mathbb{E}[\tilde{v}_{t+1}^*(\mathbf{r}(\mathbf{y}, \xi_t))] + \alpha(1 - \theta)K(\mathbf{r}(\mathbf{y} - \mathbf{x}))] \\
&\geq W_t(\theta \mathbf{x} + (1 - \theta)\mathbf{y}) + \alpha \mathbb{E}[\tilde{v}_{t+1}^*(\theta \mathbf{r}(\mathbf{x}, \xi_t) + (1 - \theta)\mathbf{r}(\mathbf{y}, \xi_t))] \\
&= W_t(\theta \mathbf{x} + (1 - \theta)\mathbf{y}) + \alpha \mathbb{E}[\tilde{v}_{t+1}^*(\mathbf{r}(\theta \mathbf{x} + (1 - \theta)\mathbf{y}, \xi_t))] \\
&= G_t(\theta \mathbf{x} + (1 - \theta)\mathbf{y}).
\end{aligned}$$

The first inequality is due to W_t being convex and $\mathbf{r}(\mathbf{y}, \xi) - \mathbf{r}(\mathbf{x}, \xi) = (\sum_{i=1}^n (y_i - x_i), 0, \dots, 0)$ (Noting that $K(y_1 - x_1, \dots, y_n - x_n) \geq K(\sum_{i=1}^n (y_i - x_i), 0, \dots, 0)$ since $K_n > K_{n-1} > \dots > K_1$), the second inequality is due to the $\alpha \mathbf{K}$ -convexity of $\alpha \tilde{v}_{t+1}^*$ (See Lemma 3), and finally the second equality is due to the fact we use a linear mapping $\mathbf{r}(\mathbf{x}, \xi) = (\sum_{i=1}^n x_i - \xi, 0, \dots, 0)$.

Given some vectors $\mathbf{x}^1 = (x_1^1, \dots, x_n^1) \leq \mathbf{x}^2 = (x_1^2, \dots, x_n^2)$, let $\mathbf{y}_t(\mathbf{x})$ be the optimal order up to level under state \mathbf{x} . Suppose that $\mathbf{y}_t(\mathbf{x}^1) - \mathbf{x}^1 = w_i \mathbf{e}_i$ for some i . Then we can choose $\hat{\mathbf{x}}^2$ such that $\hat{\mathbf{x}}^2 = \mathbf{x}^1 + \sum_{i=1}^n (x_i^2 - x_i^1) \mathbf{e}_i$. Again, it follows that $\tilde{v}_t^*(\hat{\mathbf{x}}^2) = \tilde{v}_t^*(\mathbf{x}^2)$ by the virtue of Lemma 1.

First, we consider the case that $\mathbf{y}_t(\mathbf{x}^1) \geq \theta \mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2$. We have

$$\begin{aligned}
& \theta[\tilde{v}_t^*(\mathbf{x}^1)] + (1 - \theta)[\tilde{v}_t^*(\hat{\mathbf{x}}^2) + K(\hat{\mathbf{x}}^2 - \mathbf{x}^1)] \\
&= \theta[G_t(\mathbf{y}_t(\mathbf{x}^1)) + K(\mathbf{y}_t(\mathbf{x}^1) - \mathbf{x}^1)] + (1 - \theta)[G_t(\mathbf{y}_t(\hat{\mathbf{x}}^2)) + K(\mathbf{y}_t(\hat{\mathbf{x}}^2) - \hat{\mathbf{x}}^2) + K(\hat{\mathbf{x}}^2 - \mathbf{x}^1)] \\
&\geq \theta[G_t(\mathbf{y}_t(\mathbf{x}^1)) + K(\mathbf{y}_t(\mathbf{x}^1) - \mathbf{x}^1)] + (1 - \theta)[G_t(\mathbf{y}_t(\hat{\mathbf{x}}^2)) + K(\mathbf{y}_t(\hat{\mathbf{x}}^2) - \mathbf{x}^1)] \\
&\geq \theta[G_t(\mathbf{y}_t(\mathbf{x}^1)) + K(\mathbf{y}_t(\mathbf{x}^1) - \theta \mathbf{x}^1 - (1 - \theta)\hat{\mathbf{x}}^2)] + (1 - \theta)[G_t(\mathbf{y}_t(\hat{\mathbf{x}}^2)) + K(\mathbf{y}_t(\hat{\mathbf{x}}^2) - \theta \mathbf{x}^1 - (1 - \theta)\hat{\mathbf{x}}^2)] \\
&\geq G_t(\mathbf{y}_t(\theta \mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2)) + K(\mathbf{y}_t(\theta \mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2) - \theta \mathbf{x}^1 - (1 - \theta)\hat{\mathbf{x}}^2) \\
&= \tilde{v}_t^*(\theta \mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2).
\end{aligned}$$

The first inequality is due to the subadditivity property of $K(\mathbf{x})$, the second inequality is due to $K(\mathbf{x})$ being component-wise non-decreasing in \mathbf{x} , and the third inequality is due to the fact that among all $\mathbf{y} \geq \theta \mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2$, $\mathbf{y}_t(\theta \mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2)$ minimizes $G_t(\mathbf{y}) + K(\mathbf{y} - \theta \mathbf{x}^1 - (1 - \theta)\hat{\mathbf{x}}^2)$ and

by our design we know that $\mathbf{y}_t(\mathbf{x}^1) \geq \theta\mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2$ and $\mathbf{y}_t(\hat{\mathbf{x}}^2) \geq \hat{\mathbf{x}}^2 \geq \theta\mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2$.

Next, we consider the case where $\mathbf{y}_t(\mathbf{x}^1) < \theta\mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2$, which means that $\mathbf{y}_t(\theta\mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2) = \theta\mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2$ by Lemma 1 and Theorem 3. Note that $G_t(\mathbf{x}^1) > G_t(\mathbf{y}_t(\mathbf{x}^1)) + K_i$ and $\mathbf{y}_t(\mathbf{x}^1) = \mathbf{x}^1 + w_i\mathbf{e}_i$ and $\hat{\mathbf{x}}^2 = \mathbf{x}^1 + \sum_{i=1}^n (x_i^2 - x_i^1)\mathbf{e}_i$. By the continuity of $G_t(\mathbf{x})$, there exists $\tilde{\mathbf{x}} = \mathbf{x}^1 + \tilde{w}_i\mathbf{e}_i$ for some $0 < \tilde{w}_i < w_i$ such that $G_t(\tilde{\mathbf{x}}) = G_t(\mathbf{y}_t(\mathbf{x}^1)) + K(\mathbf{y}_t(\mathbf{x}^1) - \mathbf{x}^1)$. Hence there must exist $\rho \geq \theta$ such that $\rho\tilde{\mathbf{x}} + (1 - \rho)\hat{\mathbf{x}}^2 = \theta\mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2$ (since $\tilde{\mathbf{x}} - \mathbf{x}^1 = \tilde{w}_i\mathbf{e}_i$ and $\tilde{w}_i > 0$). By \mathbf{K} -convexity in \mathbb{R}^n for G_t , we have

$$G_t(\theta\mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2) = G_t(\rho\tilde{\mathbf{x}} + (1 - \rho)\hat{\mathbf{x}}^2) \leq \rho G_t(\tilde{\mathbf{x}}) + (1 - \rho)[G_t(\hat{\mathbf{x}}^2) + K(\hat{\mathbf{x}}^2 - \tilde{\mathbf{x}})].$$

But since we have $K(\hat{\mathbf{x}}^2 - \tilde{\mathbf{x}}) = K(\hat{\mathbf{x}}^2 - \mathbf{x}^1) = K_i$ and $G_t(\hat{\mathbf{x}}^2) \geq G_t(\mathbf{y}_t(\mathbf{x}^1))$ (Otherwise, $\mathbf{y}_t(\mathbf{x}^1)$ cannot be optimal), then we have

$$\begin{aligned} & \theta[\tilde{v}_t^*(\mathbf{x}^1)] + (1 - \theta)[\tilde{v}_t^*(\hat{\mathbf{x}}^2) + K(\hat{\mathbf{x}}^2 - \mathbf{x}^1)] \\ &= \theta[G_t(\mathbf{y}_t(\mathbf{x}^1)) + K(\mathbf{y}_t(\mathbf{x}^1) - \mathbf{x}^1)] + (1 - \theta)[G_t(\hat{\mathbf{x}}^2) + K(\hat{\mathbf{x}}^2 - \mathbf{x}^1)] \\ &= \theta[G_t(\tilde{\mathbf{x}})] + (1 - \theta)[G_t(\hat{\mathbf{x}}^2) + K(\hat{\mathbf{x}}^2 - \mathbf{x}^1)] \\ &\geq \rho[G_t(\tilde{\mathbf{x}})] + (1 - \rho)[G_t(\hat{\mathbf{x}}^2) + K(\hat{\mathbf{x}}^2 - \tilde{\mathbf{x}})] \\ &\geq G_t(\rho\tilde{\mathbf{x}} + (1 - \rho)\hat{\mathbf{x}}^2) \\ &= \tilde{v}_t^*(\theta\mathbf{x}^1 + (1 - \theta)\hat{\mathbf{x}}^2). \end{aligned}$$

The first inequality is due to the fact that $G_t(\mathbf{y}_t(\mathbf{x}^1)) + K(\mathbf{y}_t(\mathbf{x}^1) - \mathbf{x}^1) \leq G_t(\hat{\mathbf{x}}^2) + K(\hat{\mathbf{x}}^2 - \mathbf{x}^1)$ and $\rho \geq \theta$, and the second inequality is due to the \mathbf{K} -convexity of G_t .

The continuity of $\tilde{v}_t^*(\mathbf{x})$ is due to the fact that we know that by Theorem 2, v_t^* is continuous in x , and the equivalence between the original dynamic programming formulation and the n -dimensional dynamic programming formulation.

Once we know that G_t is \mathbf{K} -convex in \mathbb{R}^n , then by Theorem 3, it follows that the generalized (s, S) policy is optimal. ■

Remark 1 *As shown in Gallego and Sethi (2005), \mathbf{K} -convexity in \mathbb{R}^n generally is not preserved*

under minimization. Here we utilize the property of concave ordering cost and Lemma 1, i.e., the ordering decisions only depend on $\sum_{i=1}^n x_i$. This is generally not true for other multi-dimensional dynamic programming formulations.

It is clear that if we consider an ordering cost consisting of a setup cost and a linear ordering cost, then we have the same result as in Scarf (1958), although our proof procedure for that case is slightly different from Scarf (1958).

Next, suppose now that the ordering cost $c(z)$ in each period is a general increasing concave function of z . Then we can show that the optimal order-up-to level $y_t(x)$ is a non-increasing function of the current inventory position x . This result follows from the fact that we can approximate an increasing concave function by a piecewise linear concave function with arbitrary accuracy. Thus, our results in Theorem 1 and Lemma 2 apply to systems with general concave ordering costs.

6 Extensions to Other Settings

In this section, we briefly explain how our approach can be extended to time varying ordering costs, the lost sales case and the non-zero leadtime case with backordering.

First, our result can also be extended to the following time varying piecewise concave ordering costs: $c_t(x) = \min_i \{K_{i,t}\delta(x) + c_{i,t}x\}$ with $\alpha K_{i,t+1} \leq K_{i,t}$ (We can make use of property (3) in Lemma 3.) for all $i = 1, \dots, n$ and all t . In this case we can reformulate the expected discounted cost as

$$\begin{aligned} & \mathbb{E} \left\{ \sum_{t=1}^T \alpha^t \left[\sum_{i=1}^n K_{i,t} \delta(y_{i,t} - x_{i,t}) + \sum_{i=1}^n c_{i,t} (y_{i,t} - x_{i,t}) + g_t \left(\sum_{i=1}^n y_{i,t} - \xi_t \right) \right] \right\} \\ &= \sum_{t=1}^T \alpha^t \left[\sum_{i=1}^n K_{i,t} \delta(y_{i,t} - x_{i,t}) + W_t(\mathbf{y}_t) \right] - \alpha \sum_{i=1}^n c_{i,1} x_{i,1}, \end{aligned}$$

where $W_t(\mathbf{y}_t) = \sum_{i=1}^n c_{i,t} y_{i,t} + \mathbb{E}[g_t(\sum_{i=1}^n y_{i,t} - \xi_t) - \alpha \sum_{i=1}^n c_{i,t+1} r_i(\mathbf{y}_t, \xi_t)]$. Then we can show that the structure of the optimal policy is also a generalized (s, S) policy by a similar argument as in the stationary concave ordering cost case.

Next, we consider the lost sales case with a piecewise linear concave ordering cost $c(\cdot)$. Let

$p_t(x)$ be the lost sales cost if there are x unsatisfied demand in period t . In this case, the function G_t is modified as follows

$$G_t(y) = \mathbb{E}[h_t([y - \xi]^+) + p_t([\xi - y]^+)] + \alpha \mathbb{E}v_{t+1}^*([y - \xi]^+). \quad (13)$$

Let $v_t^*(x)$ be the value function in period t when the inventory level in period t is x . Note that here we only allow $x \geq 0$, i.e., x cannot be negative. Then the optimality equation is given by

$$v_t^*(x) = \min_{y \geq x} [c(y - x) + G_t(y)]. \quad (14)$$

Given x , let $y_t(x)$ be the smallest minimizer of $c(y - x) + G_t(y)$, i.e.,

$$y_t(x) = \min \arg \min_{y \geq x} \{c(y - x) + G_t(y)\}. \quad (15)$$

Note that the local monotone property still holds. If h_t and p_t are convex functions, we can show a similar \mathbf{K} -convex property for the n -dimensional equivalent dynamic programming formulation. It follows that we still have Condition 1 for the optimal policy and the generalized (s, S) policy is optimal, although the generalized (s, S) could be truncated due to the fact that the inventory level is always non-negative.

Finally, we consider the case with fixed leadtime l , where an order placed in period t is delivered in period $t+l$. Let x be the current inventory in stock, and x_i be the amount of inventory delivered i periods later, where $i = 1, \dots, l-1$. Let $v_t^*(x, x_1, \dots, x_{l-1})$ be the corresponding value function, then the optimality equation is given by

$$v_t^*(x, x_1, \dots, x_{l-1}) = \min_{z \geq 0} \{c_t(z) + L_t(y) + \alpha \mathbb{E}[v_{t+1}^*(x + x_1 - \xi_t, x_2, \dots, z)]\}, \quad (16)$$

where

$$L_t(y) = \mathbb{E}[h_t([y - \xi]^+) + b_t([\xi - y]^+)].$$

We can show that v_t^* can be re-expressed as

$$\begin{aligned}
v_t^*(x, x_1, \dots, x_{l-1}) &= L_t(x) + \alpha \mathbb{E}L_{t+1}(x + x_1 - \xi_{t+1}) + \dots \\
&\quad + \alpha^{l-1} \mathbb{E}L_{t+l-1}(x + \dots + x_{l-1} - \sum_{i=t}^{t+l-1} \xi_k) \\
&\quad + q_t(x + x_1 + \dots + x_{l-1}),
\end{aligned} \tag{17}$$

where $q_t(u)$ is defined recursively by

$$q_t(u) = \min_{y \geq u} \{c_t(y - u) + \alpha \mathbb{E}L_{t+l}(y - \sum_{i=t}^{t+l} \xi_k) + \alpha \mathbb{E}q_{t+1}(y - \xi_{t+1})\}. \tag{18}$$

With $q_t(u)$ we can carry out the same analysis as in the zero leadtime case.

7 Conclusion

In this paper we demonstrate the optimality of generalized (s, S) policies for a broad class of inventory systems with piecewise linear concave ordering costs for general demand distributions. Previous models in the literature require the restrictive assumptions that demand distributions must be Polya, uniform or log-concave. Hence, we resolve the open question as to whether the generalized (s, S) policies are optimal for general demand distributions.

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