Similitude and Scaling in Product Design Analysis

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I. Introduction

The process of similitude and scaling is an important step in the evaluation of a product due to its ability to conserve time and cost by utilizing the resources efficiently to predict performance of the product for different input conditions. The procedure of Traditional Similitude Method (TSM), employing Buckingham's π theorem is a classical technique used in scaling theory while Empirical Similitude Method (ESM) is a more recent technique developed to overcome some of the shortcomings of TSM and enhance the process of prediction to incorporate the influence of material nonlinear characteristics and geometric distortions. This technical report compares the two techniques in different domains for a set of test configurations.

[Dutson, 2002] developed the experimental procedure to evaluate certain pre-determined performance characteristics of a product through the use of intermediate test specimens (which are described in more detail in later sections of this report) while [Szirtes, 2003] provided a mathematical procedure for an analytical estimation of the same performance characteristics in an identical product. The intent of this description is to compare the two procedures for compatibility and correctness by providing yet another numerical verification through Finite Element Analysis (FEA) analysis performed in Solid Works® modeling and analysis package.

II. Comparison and Correlation

This section explains the working procedure and comparison results of both techniques for each of the configurations detailed below.

Configuration 1: **<u>A Simple Cantilever Beam</u>**

The beam is a stainless steel beam with dimensions 1.6 × 0.1 × 0.04. A load of 1000N is applied to the free end and a measure for the lateral displacement in the vertical direction is obtained. The geometry being simple, the verification is only done in numerical and analytical domains. Figure 1. shows the geometry of the structure with loads and boundary conditions along with the meshed structure. The analytical method is explained below.

The lateral deflection f of a cantilever beam at the free end is given by

$$f = \frac{PL^3}{3EI} \tag{1}$$

where P is the magnitude of load applied, L is the length of the specimen, E is the Young's modulus and I is the moment of Inertia. Further, the dimensionless π parameters [Szirtes, 2003] for a cantilever beam are given by

$$\pi_1 = \frac{f}{L}, \pi_2 = \frac{P}{L^2 E}, \pi_3 = \frac{I}{L^4} \tag{2}$$

such that

$$\pi_1 = \frac{\pi_2}{3\pi_3} \tag{3}$$

Given that the Young's modulus E of steel is 2×10^{11} N/m^2 and Moment of Inertia $I = ab^3/12 = 5.333 \times 10^{-7}$ m^4, the free lateral deflection is

$$f = \frac{1000 \times 1.6^3}{3 \times (5.333 \times 10^{-7}) \times (2 \times 10^{11})} = 0.0128m \tag{4}$$

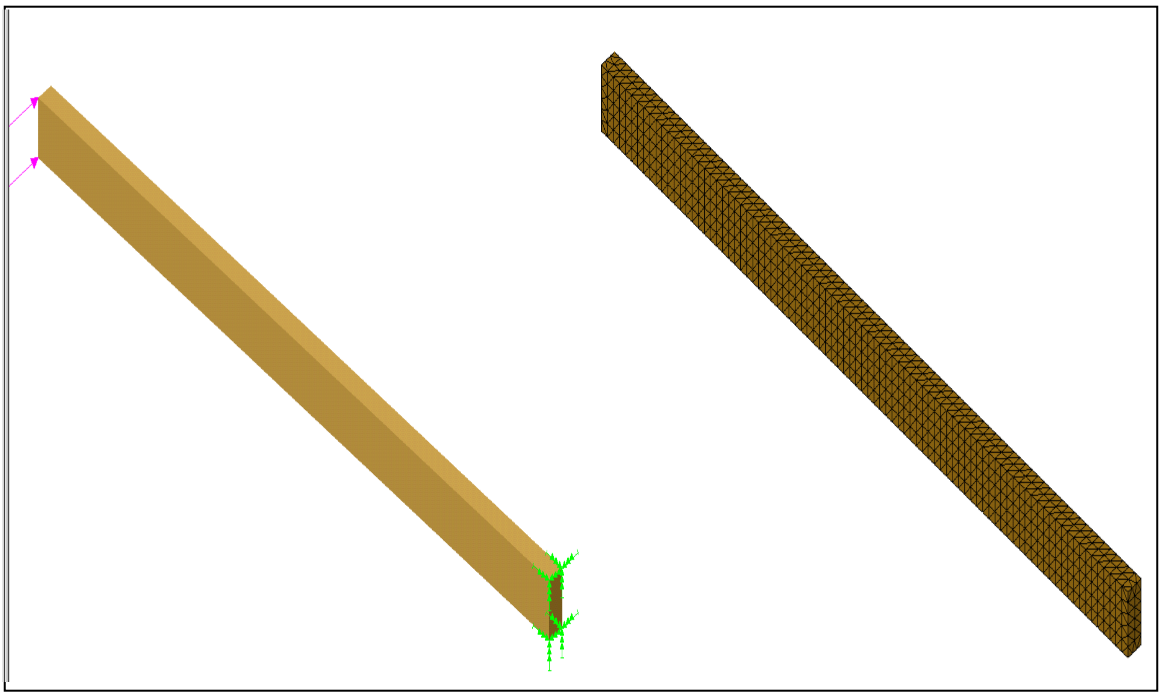

Figure 1. Geometry, Loads and Boundary Conditions and Meshed Solid

The FEA analysis was performed with medium size mesh and the results are plotted below.

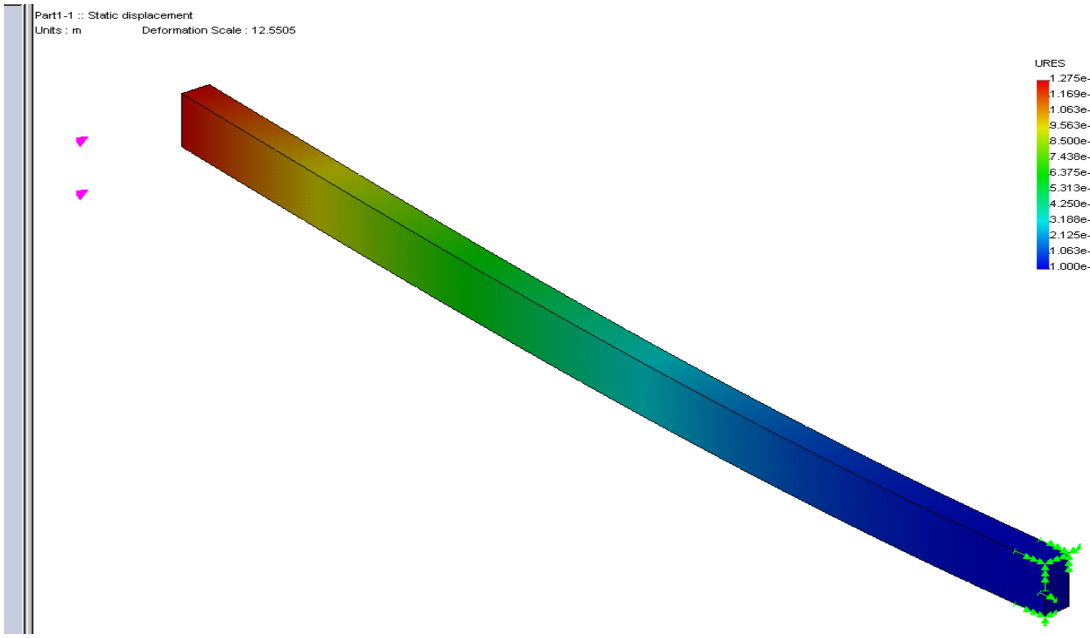

Figure 2. Static Deflection of the Cantilever Beam

The value of static lateral deflection at the free end from the FEA model and analysis is

$$f_{FEA} = 1.275 \times 10^{-2} m \tag{5}$$

The red zone in Figure 2. indicates the region of the solid under largest deflection with decreasing magnitude towards the fixed end, indicated by the blue region. Comparing values (4) and (5), a positive assertion can be made that the analytical approach to determine the free lateral deflection through the use of dimensional analysis is a feasible technique. The magnitude of error is 0.39%. The method is now extended to a solid with geometric distortion and further, experimental verification is also included in the analysis for the next configuration.

Configuration 2: <u>A Simple Cantilever Beam with Holes that are uniformly placed</u>

In this design, the beam is still prismatic with the exception that holes are placed uniformly along the length of the beam. [Szirtes, 2003] suggests that the free lateral deflection of such a beam can still be determined using TSM when certain changes are made to the evaluation procedure even though there is a distortion in the system. [Cho, 1999] and [Dutson, 2002] suggested that such a system is beyond the estimation limits of TSM and have hence proposed and recommended the use of ESM to experientially determine free lateral deflection through the use of a model beam, a model specimen beam and a product specimen beam. The mathematical correlation of the behavior of the three "*states*" can now be used to predict the free lateral deflection of a product beam. The material and geometric similarities are captured and scaled so that the product response is experimentally determined in contrast to an analytical approach. As a summary, the procedure and the symbols introduced and discussed by [Szirtes, 2003] are put forward for elucidation.

Table 1. Symbols and Definitions

S.No	Symbol	Definition	Relevance
1	D	Diameter of the holes	Gives a numerical estimate and working boundary for the analysis
2	J	Hole Density factor	A dimensionless parameter indicating the density of holes
3	q (= D/j)	Distance between holes in conjunction with j	Numerically shows the distance between holes in terms of their diameter D
4	L	Length of the Beam	The total working length of the beam
5	n_n	Nominal number of holes	The number of holes of diameter D theoretically possible for a given length L
6	n_a	Actual number of holes	The number of holes of diameter D physically possible for a given length L to maintain continuum
7	q_0	Nominal Distance from the ends	The Minimum distance to be maintained between the ends and their respective nearest holes

The formulae developed by [Szirtes, 2003] for the nominal and actual number of holes (n_n and n_a) along with the nominal distance (q_0) are provided for ease in comprehension.

$$n_n = \frac{Lj + D}{D(1 + j)} \tag{6}$$

$$n_a = \lfloor n_n + \lceil n_n - \lfloor n_n \rfloor - 1 \rceil \rfloor \tag{7}$$

$$q_0 = \frac{Lj + D - Dn_a(1 + j)}{2j} \tag{8}$$

where $\lfloor\ \rfloor$ and $\lceil\ \rceil$ indicate the floor and the ceiling function respectively. The dimensionless π parameters for the system are now modified to:

$$\pi_1 = \frac{f}{a}, \pi_2 = \frac{P}{a^2 E}, \pi_3 = \frac{D}{a}, \pi_4 = \frac{b}{a}, \pi_5 = \frac{L}{a} \tag{9}$$

where a and b are width and thickness of the beam. The modified deflection is given by

$$f = \frac{PL^3}{3EI} = \frac{4PL^3}{ab^3 E} \tag{10}$$

and the modified π grouping is given by

$$\pi_1 = 4\pi_2 \left(\frac{\pi_5}{\pi_4} \right)^3 \tag{11}$$

Since the equation does not incorporate the effect of the parameter π_3, the equation is modified and corrected to

$$\pi_1 = \psi \pi_2 \left(\frac{\pi_5}{\pi_4} \right)^3 \tag{12}$$

where

$$\psi = k_1 + k_2 Tan\{k_3 \pi_3\} \tag{13}$$

where $k_{1,}$ k_2 and k_3 are constants to be determined. The ψ function is an asymptotic approach to dimensional analysis and it provides an estimate for the vertical asymptote. Please refer to [Szirtes, 2003] for the derivation and explanation. For a prismatic beam with a geometry L = 2.4m, a = 0.18m, b = 0.032m, D = 0.099m and j = 0.85, we have

$$q = 0.11647m, n_n = 11.6790, n_a = 11, q_0 = 0.073147 \tag{14}$$

The values of the constants k_1, k_2 and k_3 are given by

$$k_1 = 4, k_2 = \left| \frac{\psi_1 - 4}{Tan\{\frac{\pi\pi_{31}}{2}\}} \right|_{\psi_1} = 1.281121, k_3 = \frac{\pi}{2} \tag{15}$$

and the free lateral deflection is found to be

$$f = 0.0773m \tag{16}$$

This value is now verified by an FEA model. The material of the beam is still steel while the changes are incorporated in the geometry such that holes of diameter D = 0.099m are uniformly placed along the entire length of the beam in conjunction with the constraints of equation (13). The value of the load is changed to P = 1200N force. Since a point load was difficult to introduce and apply in the FEA model, a minor adjustment was made to apply a line force on an infinitesimally small circle at the edge of the free end.