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Using Analytical Transformations for Designing Engineering Systems

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ABSTRACT

Scaled models often used for observing the engineering characteristics of a hypothetical design are limited in their applicability to linear and non – distorted systems. Certain changes have to be employed in the scaling procedure to account for material irregularities and geometric discontinuities when such situations when non – linear variations need to be accounted for. Building on the development of the Empirical Similitude Method (ESM), the technique of Conformal Mapping coupled with the use of analytic transformations provides an excellent opportunity to enhance the general method in complex domain as is detailed below.

1. INTRODUCTION:

A brief introduction is provided for elucidation purposes before proceeding towards a more rigorous mathematical treatment of the procedure. Empirical Similitude Method is a procedure developed to illustrate and capture the scaling parameters quantitatively when non-linear behavior is encountered in material properties, non – uniform input conditions exist, multi – material matrices are subjected to mechanical inputs, or when specimens' distortions are studied.

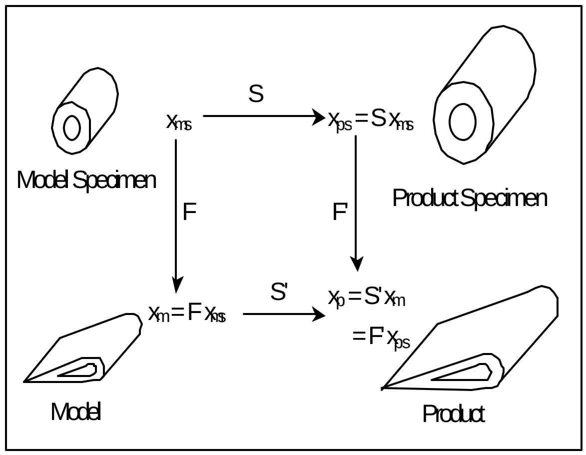

Figure 1. Empirical Similarity Method, Adapted from [Cho, 1999]

The concept of ESM establishes a correlation between the model and the product through empirical testing as illustrated in Figure 1 [Dutson, 2002]. Instead of using a direct mapping between the model and the product (which is not always accurate due to the above mentioned reasons), the procedure employs the use of intermediate specimens (as test specimens) called the model and the product specimen, which are indicative of the model and the product respectively. The model specimen, which is a geometrically simplified version of the model, is made from the same material and the manufacturing process as the model and the product specimen, which is a geometrically simplified version of the product, is made from the same material and the manufacturing process as the product.

The relationship between any two vectors can be given by a fully populated transformation matrix, the use of which forms the basis of ESM. The test data of the model and the model specimen is used to define a *scale transformation matrix S* that maps the information in the shape or geometric domain and the test data of the model specimen and the product specimen is used to generate a *form transformation matrix F* that maps the material information [Dutson, 2002]. Hence the product of these two matrices along with the model (model specimen) state vector should in principle give the response of the product.

This is possible since the ESM assumes the shape and the form transformation matrices to be independent of each other. Further, the model and the product specimens are representative of the model and the product respectively. Hence the product of the scale and form transformation matrices is assumed to be equal throughout and unique to a given system. Therefore,

$$\boldsymbol{S} \times \boldsymbol{F} = \boldsymbol{S'} \times \boldsymbol{F'} \text{ and } \boldsymbol{S} = \boldsymbol{S'},\ \boldsymbol{F} = \boldsymbol{F'}$$

An important observation is that the necessity to identify and quantify the variables that affect the system is minimized as the numerical value of the test results conveys the required information rather than the values of the influencing parameters.

2. CONFORMAL MAPPING:

The idea of designing a system using analytical transformations needs explanation. Under the regular pattern of engineering design, an "idea" is formulated and transformed thereafter to a device using certain set rules of design criteria. Put in simple terms, the "idea" is given a shape and associated material(s) form(s) are investigated to optimize cost and performance constraints while satisfying the original operational requirements. In most engineering problems encountered, real analysis of the product is initiated at an advanced stage in the product development cycle, when the product is structured with geometry and a material combination. Finite Element Analysis (FEA) packages have cut this "lead" time by providing a means to evaluate the design at an earlier stage by simulating the operating conditions. These software modules even allow for certain kinds of non-linear behavior analysis for a variety of material forms. However, ESM employs the test information of intermediate specimens as a source of data for refining the original idea, analogous to computer simulated information, with the notable exception of mechanical experiential experimentation in place of a digital evaluation.

The data thus generated needs compilation in the mathematical domain such that a transformation is generated to correlate the test information with an actual prediction of the product behavior. Conformal Mapping is a technique that is used to achieve this purpose. Analytic transformations in terms of mathematical formulae are investigated to obtain a single or a composite map such that a net system transformation is obtained.

Conformal Mapping is a process where angles and distances are mapped and their values preserved from objects to images or from the z – domain to the w – domain as is usually referred to. When the specimens are subjected to mechanical loads, certain potential fields are generated such as temperature (can be an input as well), force fields (stresses are produced) which are generally vector fields. The idea is to now capture the net effect at every point and transform it from a simple inexpensive structure to a hypothetical design.

Hence, a geometric interpretation of specimen domain and the model is generated when analytic transformations are used. This information carries the geometric data under loading conditions. On a similar note, material information is transformed as well using these analytic transformations using geometric variables. The need for analyticity is encountered when "smooth" continuous functions are desired for transformations.

3. MATHEMATICAL DEVELOPMENT:

Having briefly described the process of conformal mapping, a more formal approach is now presented to convey the idea of empirical design procedure. Following an initial study of the linear, the exponential – logarithmic and the non linear transformations, an elementary understanding of the scaling procedure was developed. The Möbius transformation is now discussed initially followed by a brief explanation on the Schwarz-Christoffel transformation to further augment the procedure.

3.1 Möbius Transformation:

The Möbius Transformation is a bilinear transformation with a general requirement of non – singular behavior. Mathematically, it is defined as:

$$w = \frac{az+b}{cz+d},\ ad\text{-}bc \neq 0$$

where c and d are never zero together. Further, the condition ad - bc = 0 characterizes critical points in the transformation. This transformation is typified by *one – to – one* correspondence of all points in the two planes and maps circular regions to scaled circular regions (preserves the circular form), a property important in ESM with regards to transformation of geometry and boundary conditions. Further, the transformation causes the cross-ratio to hold across all points in the domain, i.e., for any three points in the z-plane and the w-plane we have:

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}$$

Mathematically, the cross ratio is said to be invariant as the transformed points are fixed and unique for a given set of initial points. Modifying the cross ratio to adapt to ESM, we have:

$$\frac{(w_{ps,1} - w_{ps,2})(w_{ps,3} - w_{ps,i})}{(w_{ps,1} - w_{ps,i})(w_{ps,3} - w_{ps,2})}$$

$$= \frac{(z_{ms,1} - z_{ms,2})(z_{ms,3} - z_{ms,i})}{(z_{ms,1} - z_{ms,i})(z_{ms,3} - z_{ms,2})} \Leftrightarrow [S]$$

$$\frac{(w_{m,1} - w_{m,2})(w_{m,3} - w_{m,i})}{(w_{m,1} - w_{m,i})(w_{m,3} - w_{m,2})}$$

$$= \frac{(z'_{ms,1} - z'_{ms,2})(z'_{ms,3} - z'_{ms,i})}{(z'_{ms,1} - z'_{ms,i})(z'_{ms,3} - z'_{ms,2})} \Leftrightarrow [F]$$

$$\frac{(w_{p,1} - w_{p,2})(w_{p,3} - w_{p,i})}{(w_{p,1} - w_{p,i})(w_{p,3} - w_{p,2})}$$

$$= \frac{(z''_{ms,1} - z''_{ms,2})(z''_{ms,3} - z''_{ms,i})}{(z''_{ms,1} - z''_{ms,i})(z''_{ms,3} - z''_{ms,2})} \Leftrightarrow [S \times F]$$

$$\frac{(w_{p,1} - w_{p,2})(w_{p,3} - w_{p,i})}{(w_{p,1} - w_{ps,i})(w_{p,3} - w_{p,2})}$$

$$= \frac{(z_{m,1} - z_{m,2})(z_{m,3} - z_{m,i})}{(z_{m,1} - z_{m,i})(z_{m,3} - z_{m,2})} \Leftrightarrow [S']$$

$$\frac{(w_{p,1} - w_{p,2})(w_{p,3} - w_{p,i})}{(w_{p,1} - w_{p,i})(w_{p,3} - w_{p,2})}$$

$$= \frac{(z'_{ps,1} - z'_{ps,2})(z'_{ps,3} - z'_{ps,i})}{(z'_{ps,1} - z'_{ps,i})(z'_{ps,3} - z'_{ps,2})} \Leftrightarrow [F']$$

where w is the image domain while the complex functions, z, z' and z'' represent the object domains. The subscripts denote various test points as is required for obtaining experimental information. The requirement is to obtain the values of the constants in the mapping using the information generated above such that:

$$w_p = \frac{az_{ms} + b}{cz_{ms} + d}$$

so that the product response is obtained. Thus, a relation is generated to estimate the product response experientially.

3.2 Schwarz - Christoffel Transformation:

The Schwarz - Christoffel transformation is an integral transformation method where the area of the *upper half* of the z-plane is mapped to the interior of a polygon (in the w-plane) whose angles and sides along with the orientation are determined by the constants in the transformation. In its simplest form, the transformation is given by:

$$w = f(z) = A + C\int^{z} \prod_{k=1}^{n-1} (V - z_k)^{a_k - 1}\, dV$$

where the a's denote the interior angles of the polygon and z_k are the vertices of the polygon. The constants A and C denote the size and position of the polygon after the transformation. The choice of the polygon depends on the mapping employed. The simplest of these polygons is a straight line when all the points are collinear and interior angles are multiples of p. The integration suggests that a variable z is chosen that traces through any curve in the domain of the image plane such that the sum of products of the along any curve in the domain, including the singularities, produces a value for the mapping. Consider a simple example of a triangle. Then the mapping function would be:

$$w = f(z) = A + C\int^{z} \prod_{k=1}^{2} (V - z_k)^{a_k - 1}\, dV$$

$$w = f(z) = A + C\int^{z} \frac{1}{(V - z_1)(V - z_2)}\, dV$$

where z_1 and z_2 are the vertices of any two edges and hence are the singularities. Therefore a minimum of three points are required for the mapping so that a geometry with well-defined structure can be established in the w-plane.

The points chosen for mapping include the singularities of the z-plane and these points are generally the points on the boundary of the system and lie along the x-axis or the real axis. Being an integration formula, the singularities are accounted for in the integration by using the Residue theorem to estimate the residue at a singularity. Further, by definition, the principal values are used in the transformation and hence all analytic transformations are single valued (true even for multi-valued functions). The transformation is robust in the sense that it never goes unbounded even when $z \to \infty$ and produces a convergent definition which is hence a single value.

In relation to ESM, the mapping is being studied such that the transformation is consistent with the initial hypothesis of material and geometry independence. The geometry is evaluated using the forward and backward maps for a non - circular shape, while the material mapping, at this point, still remains a challenge.

4. EXPERIMENTAL AND NUMERICAL VALIDATION:

FEA models are presently being constructed and initial and boundary conditions are defined on the structure to match that of the experimental specimens being used for testing purposes. The values and results from FEA will be compared with that of the mathematical models and analyzed for correlation, correctness and precision. The process will be extended to all variations in geometry and material as mentioned above.

Experimental test specimens will be fabricated using Rapid Prototyping (RP) techniques, specifically Selective Laser Sintering (SLS). Different tests such as deflection analysis for simply supported and cantilever beams, steady-state temperature analysis for solids, strain measurements and stress distributions etc will be performed and contrasted with the values suggested by the mathematical models. This analysis and data in conjunction with the numerical simulation information will assist in authenticating the technique of Conformal Mapping as the scaling procedure with innovative methodology and applicative benefits in terms of expenditure and time. The eventual outcome of the research is focused on obtaining scaling laws for as distorted systems as modeling car crash tests with remote controlled cars.

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5. REFERENCES

[1] P. W. Bridgman, Dimensional Analysis, Yale University Press, New Haven, 1931.

[2] U.Cho, K. L. Wood, "Empirical Similitude Method for the Functional Test with Rapid Prototypes", Proceedings of the Solid Freeform Fabrication Symposium, Austin TX, September, 1997, pp. 559-567, 1997.

[3] U.Cho, K. L. Wood, and R. H. Crawford, "Online Functional Testing with Rapid Prototypes: a Novel Empirical Similarity Method," Rapid Prototyping Journal, 4, No. 3, pp. 128-138, 1998.

[4] U.Cho, K. L. Wood, and R. H. Crawford., "Novel empirical similarity method for the reliable product test with rapid prototypes," Proceedings of DETC, Atlanta, GA, Sept. 13-16, 1998.

[5] U. Cho, Novel Empirical Similarity Method for Rapid Product Testing and Development, Doctoral dissertation, The University of Texas at Austin, 1999.

[6] A. J. Dutson, Functional Prototyping Through Advanced Similitude Techniques, Doctoral dissertation, The University of Texas at Austin, 2002.

[7] B. S. Grewal, Higher Engineering Mathematics, Khanna Publishers, Delhi, 1998.

[8] T. Szirtes, Applied Dimensional Analysis and Modeling, McGraw-Hill, New York, 1998.

[9] R. Schinzinger, P. A. A. Laura, Conformal Mapping: Methods and Applications, Elsevier Science Publishers B.V, 1991.

[10] R. A. Silverman, Introductory Complex Analysis, Prentice – Hall, Inc., Englewood Cliffs, N.J., 1967.

[11] P. K. Kythe, Computational Conformal Mapping, Birkhäuser, Boston, 1998.

[12] T. A. Driscoll, L. N. Trefethen, Schwarz – Christoffel Mapping, Cambridge University Press, 2002.

[13] V. I. Ivanov, M. K. Trubetskov, Handbook of Conformal Mapping with Computer – Aided Visualization, CRC Press, Inc., 1995.