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Design and Performance Evaluation using Analytical Approximations

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ABSTRACT: Empirical Similitude Method (ESM) is a technique developed to address the concerns of non – linear similitude. The process disconnects the scaling of material and shape response by assuming them to be independent to form a system transformation. Such a transformation requires modeling using three representative models instead of just one as is used in the Buckingham π theorem.

In an effort to generalize this non – linear analysis, conformal mapping is employed, Schwarz – Christoffel Transformation in particular. Complex variables are used to define the system transformation over a continuous domain to obtain the scaling laws.

1. INTRODUCTION: Performance evaluation of a hypothesized product is an imperative constituent in the design process. Iteration is performed sequentially on an initial design to improve the response characteristics until the functioning of the final product measures up to the necessary standard. This conformance is required for the product to be established as a safe and reliable form of engineering output.

While design and manufacturing have been parallel processes, the testing and evaluation phase have been restricted to the finished product. The performance parameters thus obtained have been then used to refine the original conceptual design to facilitate the process of restructuring the product evolution. This is both time and cost intensive and researchers are currently focusing and combining the design process with that of experimentation to conserve time and expenditure. The integration of the design process and engineering analysis is possible through the use of dimensional analysis such that at every stage (every iteration), the specimen is scaled to behave like the model, the response observed and the necessary changes made to either the geometry and/or the material in the specimen and then re – tested until the results can be accurately mapped to the actual product.

Rapid prototyping has allowed incorporation of some of the most intricate features into the manufacturing process thus allowing for fabrication of many complex designs and complicated geometries. While this has been traditionally restricted to development of display models and to a limited extent, in small – scale manufacture, rapid prototyping can be used as a method to integrate the design process with engineering analysis for iteration on concept models and their response to input loads. Refinement in these models is hence possible so that they meet certain pre – determined standards (safety, reliability, failure etc) by fabricating a series of intermediate specimens, testing them and using the test results to either modify or re – design the previous specimen. This sets up a sequential process of fabrication and testing thus allowing for the eventual *product* to be a reliable outcome. The procedure is hence a slight deviation from traditional design in terms of theoretical formula based design or the more widely used techniques such as FEM where a single *final* geometry is tested for engineering behavior.

2. BUCKINGHAM Π THEOREM: Lord Rayleigh and Buckingham [Bridgman, 1931] in their groundbreaking work on similarity, simplified the procedure of analyzing complex structures to a large extent with the use of π groups that used the dimensions of the influencing parameters of geometry *G* and material *M* to predict the value of the parameter *P*. In the now famous Buckingham π theorem [Szirtes, 1998], the initial conditions are set to (or normalized to) zero at the outset. A complex geometry called the *product* is to be analyzed based on the information generated using the test results of a simplified but representative structure called the *model*. Assuming a steady – state analysis and formalizing the definition,

$$P_p = f(G_p, M_p) \tag{1}$$

$$P_m = f(G_m, M_m) \tag{2}$$

If the same parameter *P* is analyzed in both geometries and the functional dependence of this parameter (on geometry and material) is the same in both cases, then in conditions of similar forcing functions and boundary conditions,

$$P_p = P_m \prod \left[\phi(G_p, G_m), \psi(M_p, M_m)\right] \tag{3}$$

where $\phi(G_p, G_m)$ and $\psi(M_p, M_m)$ are the functional forms relating the geometries and the material characteristics respectively. These forms are represented in the form of π groups such that

$$P_p = P_m \prod [\pi_1, \pi_2] \tag{4}$$

or in more general terms

$$P_p = P_m \prod [\pi_1, \pi_2, ..., \pi_n] \tag{5}$$

where *n* is the number of degrees of freedom which is determined by the number of independent variables influencing the functions $\phi(G_p, G_m)$ and $\psi(M_p, M_m)$. The *dimensional* and the *echelon* matrices are then set up to identify the π groups and derive the scales (λ) thereafter. A standard example of a fluid mechanics problem [Dutson, 2002] is shown below for elucidation. If the drag force experienced by a submerged body is the variable of interest, then

$$F_d = f(l, V, \rho, \mu) \tag{6}$$

where F_d is drag force, *l* is the characteristic length, *V* is the velocity of the submerged body, ρ is the fluid density and μ is the dynamic viscosity of the fluid. Setting up the dimensional matrix, we have,

	l	V	ρ	D	μ
M	0	0	1	1	1
L	1	1	-3	1	-1
T	0	-2	0	-2	-1

Table 1. **DIMENSIONAL MATRIX**

Since F_d is the variable of interest, selecting *l*, *V* and ρ as the *repeating variables* (variables that replace the fundamental dimensions), we have,

	l	V	ρ	D	μ
l [L]	**1**	0	0	2	1
V [LT^{-1}]	0	**1**	0	2	1
ρ [ML^{-3}]	0	0	**1**	1	1

Table 2. **ECHELON MATRIX**

The matrix thus formed, referred to as the *Echelon Matrix* is actually a matrix transformation such that the repeating variables combine to from a diagonal matrix where the element along the diagonal is **1**. This transformation causes a change in the exponent or power of the dependent variables to provide for the dimensionless π parameters. In this example, we have,

$$\pi_1 = \frac{D}{l^2 V^2 \rho} \quad ; \quad \pi_2 = \frac{\mu}{lV\rho} \tag{7}$$

As can be seen from the denominators of the two fractions, the π parameters can be expressed as functions of one another. Hence,

$$\pi_1 = f(\pi_2) \quad or \quad \frac{D}{\rho V^2 l^2} = f\left(\frac{\mu}{\rho V l}\right) \tag{8}$$

and therefore for two physically similar systems, we have,

$$\left(\frac{D}{\rho V^2 l^2}\right)_p = \left(\frac{D}{\rho V^2 l^2}\right)_m \quad if \left(\frac{\mu}{\rho V l}\right)_p = \left(\frac{\mu}{\rho V l}\right)_m \tag{9}$$

The two factors defined above physically characterize and identify the drag coefficient (C_D) and the Reynolds's Number (R_e) thereby describing the type of fluid flow. Hence,

$$C_{D,p} = C_{D,m} \;\; if \;\; R_p = R_m \tag{10}$$

Notice that the analysis has scaled the system such that it is only valid if the Reynolds's numbers are proportional which goes to show that even the boundary conditions are mapped to obtain an accurate prediction. However, several disadvantages in the process limit its usefulness. The factor ρ is innately assumed to be a constant and all other factors to be independent of any functional dependence on other variables, which constricts the validity of the process. No modifications are made to understand the behavior in say mixed flow regimes or even in a dynamic environment when variables are time limited. Further, if a transient response is desired then a separate time scale has to be used to determine how fast or how slow the model experimentation has to be done, which is not always feasible due to probable un – realistic time steps. To highlight the impetus to research, the motivation is explained in the next section with examples to elucidate on the behavior of variables under non – linearity.

3. RESEARCH MOTIVATION: A complex 3 – D body can be visualized as a set or space of well – defined points with a *functional value* attached to them. These functional values (stress, temperature, strain etc) are characteristic of the material that the body is made of and define the behavior of the structure for various input loads. In most engineering problems, analysis is pursued by making several assumptions to simplify the problem. The idea behind doing so is to 'idealize' the system so that a basic understanding is obtained about its working properties. But a rational and realistic analysis would include the effects of *non – linearity* that are inherent in most engineering systems. Such non – linearities, either in geometry or material forms have to be captured for a pragmatic evaluation of the system, and is only plausible if they can be quantified numerically as functional forms.

Visualize a body of mass M moving with a total velocity V. The amount of energy spent is simply $\frac{1}{2}MV^2$ and if another body of mass $\frac{M}{2}$ has to spend the same amount of energy, then it has to move with a total velocity of $\sqrt{2}V$ where the quantity under the radical is the scale we are interested in. But we have neither accounted for the friction loss due to wind resistance nor the losses due to kinetic friction from surface interaction, and assumed mass to be independent of time (even though the reduction is small). In short, we have analyzed a system in ideal conditions.

In most engineering examples the effect of scaling and similitude is not necessarily intuitively apparent even though it influences the problem to a great extent. A simple example is put forward here for elucidation. The co – efficient of restitution or e is a scale in the momentum of a body that has an impact over another surface. Depending on the masses of the two bodies, e in its simplest form is given as

$$e = \frac{V_{A,1} - V_{B,1}}{V_{B,2} - V_{A,2}} \tag{11}$$

where the subscripts A and B denote the participating bodies. Notice that an equal mass assumption is made in the equation. Physically, the co - efficient captures the amount of energy lost in impact and thus gives a fractional value $(0 < e < 1)$, the product of which with the original velocity determines the rebound velocity. Notice also that vector quantities of velocities are scaled to define a simple scalar value. In 3 – D systems, when velocities have different axial components, e is still defined as

$$e = \frac{\overline{V}_{A,1} - \overline{V}_{B,1}}{\overline{V}_{B,2} - \overline{V}_{A,2}} \tag{12}$$

but now it has to be evaluated at every instant of time *i.e.*, $e \equiv e(t)$ and then a final scalar value needs to be defined such that for overall time

$$< e > = \frac{1}{T}\int_0^T e(t)dt \tag{13}$$

Alternately, if we use the scalar values of the velocities themselves,

$$< e > = \left| \frac{\overline{V}_{A,1} - \overline{V}_{B,1}}{\overline{V}_{B,2} - \overline{V}_{A,2}} \right| = \frac{\left|\overline{V}_{A,1}\right| - \left|\overline{V}_{B,1}\right|}{\left|\overline{V}_{B,2}\right| - \left|\overline{V}_{A,2}\right|} \tag{14}$$

where we have already accounted for time dependence. This is an important observation as vector quantities provide for instantaneous local response while scalar estimates generate time averaged global values. Both these quantities are equally important depending on the application but migration towards scalar estimation is a proposed methodology in this report.

Consider again the aerodynamic resistance encountered by a moving car. When analyzed empirically,

$$F_d = \frac{1}{2}\rho C_D A(V_c - V_a)^2 \tag{15}$$

An interesting outcome of this simple observation is that the behavior or response of *any* mechanical system can be written in a generalized expression as

$$P = f(G, M, T) + BC + IC \qquad (16)$$

where P is the parameter of interest which is a function whose arguments are G, the geometry, M, the material characteristics BC which denotes the boundary conditions, T is the time involved in the analysis. The condition T = 0 implies, steady – state analysis with some initial and boundary conditions, the condition $T \to \infty$ implies final value analysis and the condition $T = t$ represents a more general transient response. While the former two are special cases of the transient response, the boundary and the initial conditions are known parameters that affect the end solution. It is fairly obvious as to which variables in Equation (16) constitute each of these parameters.

The main motivation thus lies in analyzing any such system when each of the parameters is in itself a function of other variables. Under the effect of influencing variables, a certain randomness is introduced in the system which causes variation to be introduced in the estimation. This variation only propagates when several such variables assume random nature.

4. EMPIRICAL SIMILTUDE METHOD: [Cho,1999] and [Dutson,2002] identified limitations mentioned before in their effort to enhance the technique to non – linear domains. Recall the generalized expression in Equation (1). They modified this expression to develop a novel method of similarity analysis. Their main hypothesis was to assume that the equation can be written as

$$P = f_1(G) * f_2(M) \qquad (17)$$

for null boundary and initial conditions, and steady – state analysis. They made the extremely convenient assumption that the geometric and the material dependence are independent of each other. [Wood, 2002] gave a more practical relation by modifying the relation to

$$P = f_1(G) * f_2(M) + R(G, M) \qquad (18)$$

where $R(G, M)$ represents the error in the system. The objective now is to minimize this value with a judicious choice of $f_1(G)$ and $f_2(M)$ so that greater accuracy can be obtained in the mapping. Notice also that the two functions are convoluted (*) and not multiplied as such a generalization is not completely true.

The restrictive conditions illustrate the limitations of TSM and therefore impose the need for an analysis tool that takes into consideration all of the working properties and constraints in a dynamic environment. Hence the development of an innovative similitude procedure called *Empirical Similitude Method* that extends the correlation between the model and the product empirically, rather than with dimensional information alone [Cho, 1999]. The Empirical Similitude Method comes into play under the following conditions:

- The two systems have parameters in their governing equations that have a non-linear behavior for a given condition.
- The material properties of the systems vary from point to point. The conditions are not uniform and constant throughout the entire range of application.
- The system under consideration is comprised of multiple materials and hence various responses and varying behaviors.
- The specimens in the study have distortions in their geometries or parameters.

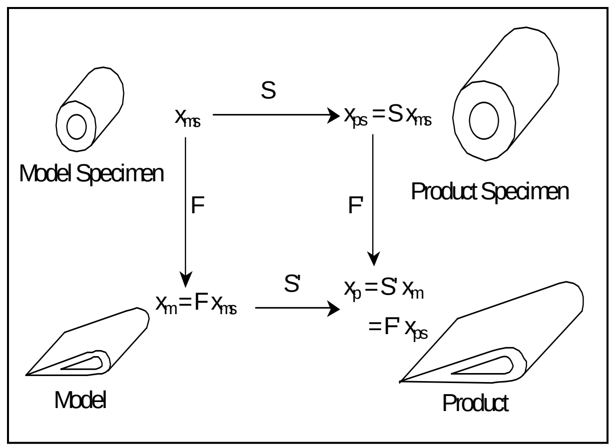

Figure 1. **ESM, ADAPTED FROM [CHO, 1999]**

The concept of ESM establishes a correlation between the model and the product through empirical testing as illustrated in Figure 1 [Dutson, 2002]. Instead of using a direct mapping between the model and the product (which is not always accurate due to the above mentioned reasons), the procedure employs the use of intermediate specimens (as test specimens) called the model and the product specimen, which are indicative of the model and the product respectively. The model specimen, which is a geometrically simplified version of the model, is made from the same material and the manufacturing process as the model and the product specimen, which is a geometrically simplified version of the product, is made from the same material and the manufacturing process as the product. If the state vectors (written in bold) of the model specimen, the model, the product specimen and the product are respectively $\boldsymbol{x_{ms}}$, $\boldsymbol{x_m}$, $\boldsymbol{x_{ps}}$, and $\boldsymbol{x_p}$, then,

$$\boldsymbol{x_p = f\,(x_{ms}, x_m, x_{ps})} \tag{19}$$

Steeping back, analyzing Equation (17) further, assume that the parameter P satisfies the Laplace equation *i.e*., it is a potential. When we make the assumption of independence, we can now proceed to solve the problem using the SOV technique of the pde that the parameter is defined as *i.e*., Laplace equation. Hence, in this simplification only a particular class of pde's (elliptic) are solved. This allows for the evaluation of the product response at every point i in its domain based on the values in the other three geometries measured at the same corresponding point. Since this is time consuming, the process of conformal mapping is used as an alternative to provide for a *field* perspective of the process, so that two or more variables can be combined for either a surface (area) or a solid (volume) evaluation.

5. CONFORMAL MAPPING: [Schinzinger et al, 1991] define conformal mapping as "*the use of functions of complex variables to transform a complicated boundary to a more readily analyzable simpler domain*". Let's look at it from the definition given in Equation (17) so that its relevance with ESM is more evident. For generality, re – writing this relation, we have

$$P(x,y,z) = G(x,y,z) * M(x,y,z) \tag{20}$$

where we have developed a *field* definition for the parameter P implying the dependence on position for its value. If we assume significant 2 – D variations alone, then the relation can be simplified using a single complex variable (w) to

$$P(x,y) = G(x,y) * M(x,y) \tag{21}$$

$$P(w) = G(w_1) * M(w_2) \tag{22}$$

The variable w has to be defined as a function continuous in its working interval. This has to be extended to 2 – D now where the complex function $f(w)$ has to be continuous in both perpendicular directions implying that it has to satisfy the Cauchy – Reimann conditions [Ahlfors,1979], or the function has to be *analytic* throughout the working domain. Further, since a *field* definition is sought using *analytic* functions, the function must also satisfy the Laplace equation or the Poisson equation (depending on the source) to generate a *potential*, the basis for using conformal mapping. While modern numerical techniques allow to circumvent some of the non – Laplacian fields [Schinzinger et al, 1991], in this paper we stick to the basic conformal mapping. So, the idea is to now use analytic functions as approximations to predict the behavior of a product under potential using equipotential definitions in the other three geometries. Primarily, z denoting complex space, we have

$$P(z) = G(z) * M(z) \tag{23}$$

As the world of complex analysis involves numerous analytic functions, we choose the standard Schwarz – Christoffel Transformation (S – C) as a test candidate to derive scaling laws.

6. SCHWARZ – CHRISTOFFEL TRANSFORMATION: The Schwarz – Christoffel transformation is an integral transformation where the upper half z – plane [Silverman, 1967] is mapped into an n – sided polygon with the limiting condition $n \geq 3$. The 'minimum' polygon that can be thus mapped into is the triangle with the interior angles measured using the transformation properties. This transformation is dual in nature implying that the function procedure can be carried out in both the forward and the backward paths to obtain the required image region. The advantage of this transformation is its ability to use the Residue theorem to evaluate the functional value at the points of singularity. This allows for estimation when the variable approaches the singularity without the function value blowing up.

The interior region of the polygon (in the w-plane) with angles and sides along with the orientation is determined by the constants in the transformation. In its simplest form, the transformation is given by

$$w = f(z) = A + C\int^{z} \prod_{k=1}^{n-1} (\varsigma - z_k)^{\alpha_k - 1} d\varsigma \tag{24}$$

where the α's denote the interior angles of the polygon and z_k are the vertices of the polygon. The constants A and C denote the position and size of the polygon after the transformation where the choice of the polygon depends on the mapping employed. Re – writing the equation for simplicity, we have

$$w = f(z) = C\int_0^{z} \prod_{k=1}^{n} (\varsigma - z_k)^{\alpha_k - 1} d\varsigma \tag{25}$$

Consider now a simple example of a mapping into a triangle. The mapping function would be

$$w = f(z) = C\int_0^z \prod_{k=1}^{3} (\varsigma - z_k)^{\alpha_k - 1} d\varsigma \qquad (26)$$

If the vertices of the triangle are z_1, z_2 and z_3 then

$$w = C\int_0^z (\varsigma - z_1)^{\alpha_1 - 1}(\varsigma - z_2)^{\alpha_2 - 1}(\varsigma - z_3)^{\alpha_3 - 1} d\zeta$$

with

$$\sum_{k=1}^{3} \alpha_k \pi = \pi \text{ and } \alpha_k \pi = n\pi, 0 < n < 1 \qquad (27)$$

If the triangle is equilateral, then

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3} \qquad (28)$$

which then gives

$$w = C\int_0^z \frac{1}{\sqrt[3]{(\varsigma - z_1)^2(\varsigma - z_2)^2(\varsigma - z_3)^2}} d\zeta \qquad (29)$$

The only term left to determine is the constant C which is evaluated using the same relation as is shown below. Integrating between any two singularities, if the length of the equilateral triangle is l, then

$$C = \frac{l}{\displaystyle\int_{z_1}^{z_2} \frac{1}{\sqrt[3]{(\varsigma - z_1)^2(\varsigma - z_2)^2(\varsigma - z_3)^2}} d\zeta} \qquad (30)$$

which then gives the required transformation

$$w = f(z) = l\left[\frac{\displaystyle\int_0^z \frac{1}{\sqrt[3]{(\varsigma - z_1)^2(\varsigma - z_2)^2(\varsigma - z_3)^2}} d\zeta}{\displaystyle\int_{z_1}^{z_2} \frac{1}{\sqrt[3]{(\varsigma - z_1)^2(\varsigma - z_2)^2(\varsigma - z_3)^2}} d\zeta}\right]$$

We extend this transformation to a square for the purpose of applying it to a stress analysis problem. Assume that we are dealing with simply connected regions *i.e.*, the region does not intersect itself more than once. Further, lets look at a standard transformation given in the S – C realm. The transformation [Silverman, 1967)

$$w = f(z) = \int_0^z \frac{dt}{\sqrt{a^4 - t^4}} \qquad (31)$$

maps a disk of radius *a* to a square of length $\sqrt{2}a$. It is straightforward to notice that

$$w = \int_0^z \{(a - t)(a + t)(t + ia)(t - ia)\}^{-\frac{1}{2}} dt \qquad (32)$$

and hence the vertices of the polygon are $(\pm a, \pm ia)$ which is a square. Further, the radical arises from the fact that $\alpha_k = \frac{1}{2} \forall\, k \in [1,4]$ since

$$\alpha_i = \alpha_j \text{ and } \sum_{i=1}^{4} \alpha_i = 2\pi \qquad (33)$$

A disk is mapped since we transform the region $|t| \leq a$ for the integrand in Equation (31) to exist and be real. Having provided the basic idea of mapping using the S – C transformation, we now proceed to apply it to the ESM and a stress analysis problem thereafter. Assume that the *product* and *model* are square plates of negligible thickness, and the *model specimen* and the *product specimen* are disks (circular plates of negligible thickness). Pure geometric mapping between the *model* and the *model specimen* (or the *product* and the *product specimen*) can be accomplished using the S - C transformation. Further, let the *product specimen* and the *model specimen* be scaled by one half to that of the *product* and *model* respectively. We have now established a well-scaled problem as is illustrated below. The *model* transformation would map the disk to a square of edge $2\sqrt{2}a$, which is twice the original value. Hence if the *model specimen* has a radius *a* and the *product specimen* 2*a*, then the squares they map into would have edge lengths $\sqrt{2}a$ and $2\sqrt{2}a$. Thus the ratios are maintained at a value of 2 for proper scaling.

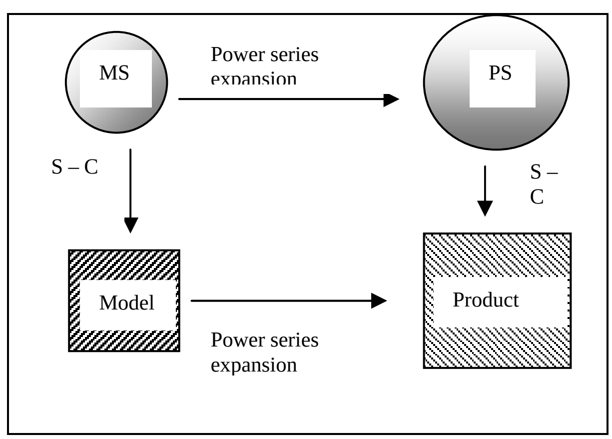

Figure 2. **CONFORMAL MAPPING APPLIED TO ESM**

Capturing the scale from the *model* to the *product* (or from the *model specimen* to the *product specimen*) as a functional form is straightforward, as a simple relation such as 2*z* would suffice for the necessary mapping parameters. However this is a geometric map and we intend to capture material properties between the *product specimen* and the *model specimen*. This is achieved in an alternate procedure discussed below as part of the introduction to the stress analysis problem.

7. STRESS ANALYSIS PROBLEM: Let the Von Mises stress of the *product* square plate made out of Aluminum be the parameter of interest. From the definition of the ESM process, the *product specimen* is a disk made from Aluminum, the *model* is a square plate made from Nylon and the *model specimen* is a disk made from Nylon. The idea is to now obtain the stress profile of the *product* based on the stress profiles of the other three geometries. We are interested in mapping stress potential lines, as they are the locus of points at the same stress potential. Let the boundaries of all geometries be fixed with a point load at the geometric center of magnitude P = 100 N. Hence for any point *i*

$$\frac{\sigma_{ps,i}}{\sigma_{ms,i}} = \Lambda_{1,i}, |r| \le a \tag{34}$$

and

$$\frac{\sigma_{p,i}}{\sigma_{m,i}} = \Lambda_{2,i}, (|x|,|y|) \le a, (|x'|,|y'|) \le 2a$$
$$\Rightarrow \sigma_{p,i} = \Lambda_{2,i}\sigma_{m,i} \tag{35}$$

Hence, for every point *i* in the region *G* in the *model* disk, Λ_2 being the required functional form for the material mapping,

$$\sigma_p = \Lambda_2 \sigma_m \tag{36}$$

From the S – C definition between the *model* and the *model specimen* letting *C* be the correction factor equal to $\left(\frac{\sigma_{m,act}}{\sigma_{m,pred}}\right)$, we have

$$\sigma_m = C\left(\int_0^z \frac{dt}{\sqrt{a^4 - t^4}}\right)\sigma_{ms} \tag{37}$$

Combining Equations (36) and (37),

$$\sigma_p = C\Lambda_2\left(\int_0^z \frac{dt}{\sqrt{a^4 - t^4}}\right)\sigma_{ms} \tag{38}$$

The evaluation of the term Λ_2 would require testing of the *product*, which would negate the basic development of ESM. To circumvent the issue, we assume that the functional form for Λ_2 is identical to that of Λ_1. This would generate an error but would eliminate the need to test the *product*. Hence,

$$\sigma_p \approx C\Lambda_1\left(\int_0^z \frac{dt}{\sqrt{a^4 - t^4}}\right)\sigma_{ms} \tag{39}$$

or E_1 being the error due to the assumption

$$\sigma_p \approx \{f_1(z) * f_2(z)\}\sigma_{ms} + E_1 \tag{40}$$

Unlike $f_2(z)$, since the form of $f_1(z)$ is unknown, we fit a power series expansion such that,

$$f_1(z) = C\Lambda_1 = \sum_{n=0}^{\infty} a_n \sigma_z^n$$
$$\Rightarrow \sigma_p \approx \{f_1(z) * f_2(z)\}\sigma_{ms} + E_1 + E_2 \tag{41}$$

The power series expansion would be evaluated to a point till the R^2 value indicates a good enough fit, thereby causing the extra error term E_2. Having set up the problem, we proceed to evaluate it using a finite element study for the stress profiles. To accommodate simplicity in presentation, we only look at points in the vertical direction starting from the geometric center of each structure such that

$$\sigma_p \approx \left(\sum_{n=0}^{m} a_n \left|\sigma_{|z|}\right|^m\right)\left(\int_0^{|z|} \frac{dt}{\sqrt{1 - t^4}}\right)\sigma_{ms} \tag{42}$$

where *m* is the degree of expansion provided by the least squares fit of the scales developed from the *product specimen* and the *model specimen* and $|z|$ is the distance in the vertical direction from the geometric center. The *model specimen* disk has a radius of 5cm normalized to 1 which then gives the normalized radius of the *product specimen* plate to be 2, the *model* would have an normalized edge of $\sqrt{2}$ and the *product*, a normalized edge of $2\sqrt{2}$. All geometries have a thickness of 1mm, which is negligible compared to the other major dimensions.

The produced stress potential lines of the three geometries apart from the *product* are shown below.

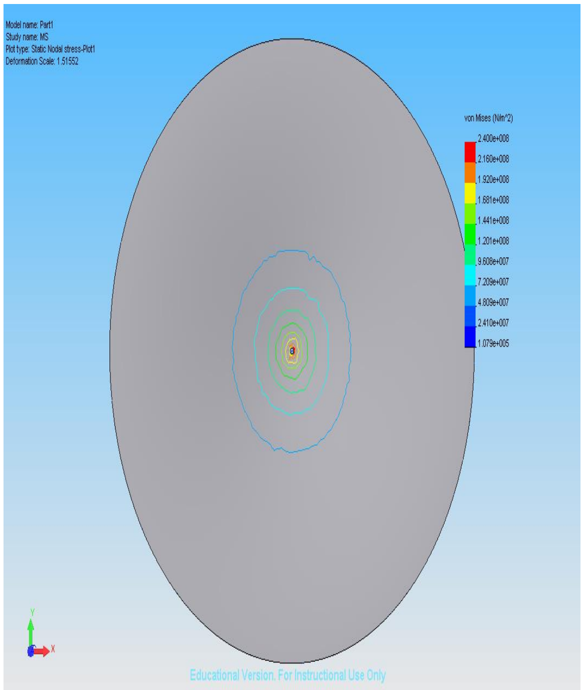

Figure 3. **STRESS ANALYIS OF THE NYLON MODEL SPECIMEN**

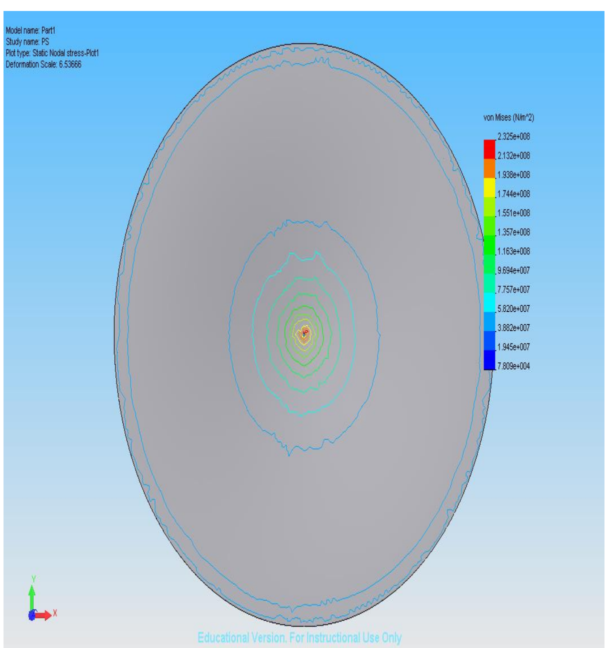

Figure 4. **STRESS ANALYIS OF THE NYLON MODEL**

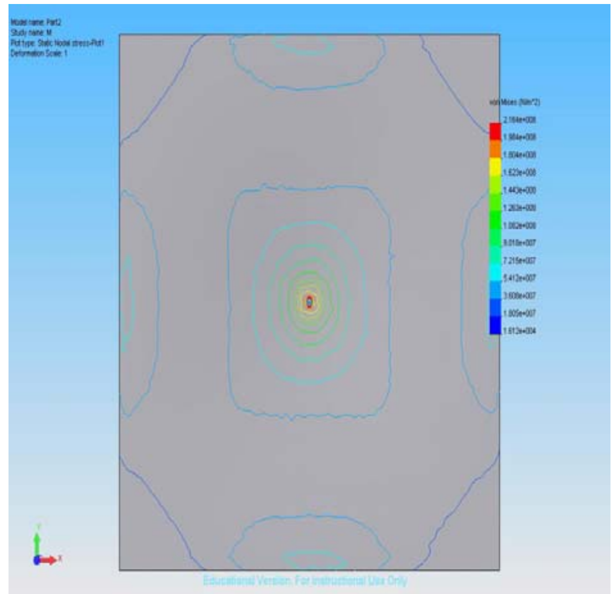

Figure 5. **STRESS ANALYIS OF THE ALUMINUM PRODUCT SPECIMEN**

Notice that the distortion in the stress lines is evident in the various geometries as the shape changes across the domain. The *model specimen* made from Nylon shows a gradual and smooth distribution of stress while it takes a different orientation in the *model*. The *product specimen* has a completely different distribution of stress values and it is a bit haphazard when compared to the *model* and the *model specimen*. The two scales mentioned earlier are combined to obtain a system scale. The fitted scale curve is shown below which required a polynomial definition of the 6th order to obtain a satisfactory R^2 value. Once the fitted scale is obtained, we proceed to predict the stress values in the *product* and the results are shown below.

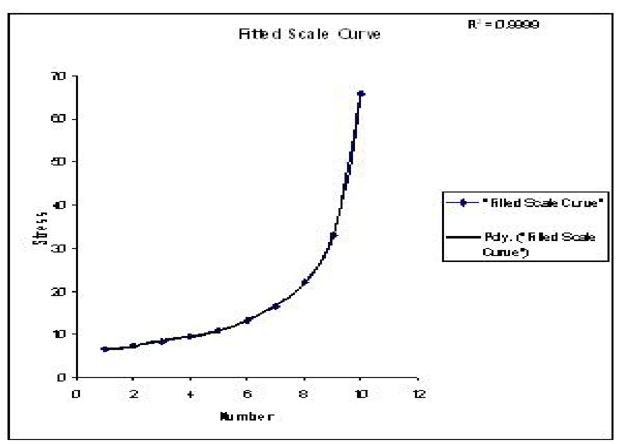

Figure 6. **FITTED SCALE CURVE**

Point	σ_{ms}	σ_{ps}	Λ_1
1	2.40E+08	2.33E+08	0.9688
2	2.16E+08	2.09E+08	0.969
3	1.92E+08	1.86E+08	0.9693
4	1.68E+08	1.63E+08	0.9685
5	1.44E+08	1.40E+08	0.9688
6	1.20E+08	1.16E+08	0.9684
7	9.61E+07	9.31E+07	0.9687
8	7.21E+07	6.98E+07	0.9685
9	4.81E+07	4.66E+07	0.9684
10	2.41E+07	2.33E+07	0.968

Table 3. **DERIVED PARAMETERS**

Model (σ_m)		Correction	
Predicted	**Actual**	**C**	$C \times \Lambda_1$
3.17E+07	2.16E+08	6.83	6.62
2.57E+07	1.95E+08	7.59	7.36
2.03E+07	1.73E+08	8.54	8.28
1.55E+07	1.52E+08	9.76	9.45
1.14E+07	1.30E+08	11.38	11
7.92E+06	1.08E+08	13.65	13.2
5.07E+06	8.66E+07	17.07	16.5
2.85E+06	6.49E+07	22.75	22
1.27E+06	4.33E+07	34.11	33
3.18E+05	2.17E+07	68.11	65.9

Table 4. **CORRECTION PARAMETERS**

Product (σ_p)			
Predicted	**Actual**	**Error**	**Error %**
2.10E+08	2.68E+08	5.80E+07	21.66
1.89E+08	2.41E+08	5.23E+07	21.71
1.68E+08	2.14E+08	4.65E+07	21.708
1.47E+08	1.88E+08	4.09E+07	21.789
1.26E+08	1.61E+08	3.50E+07	21.739
1.05E+08	1.34E+08	2.92E+07	21.809
8.39E+07	1.07E+08	2.33E+07	21.765
6.29E+07	8.04E+07	1.75E+07	21.753
4.19E+07	5.36E+07	1.17E+07	21.755
2.10E+07	2.68E+07	5.82E+06	21.732

Table 5. **PRODUCT STRESS – PREDICTED AND ACTUAL**

8. ERROR ANALYSIS: As is seen from the table above, the errors produced are very uniform and consistent even though their magnitude is on the higher side. This shows that the errors are predictable as well in the process. The main sources of these errors are the assumptions mentioned before. A detailed explanation is provided to highlight the origin and the contribution of the individual geometric and material properties.

The geometric transformation carried out as given by the $S - C$ transformation is exact in the sense that every point in the object space *i.e.*, the disk is uniquely transformed in to the image space *i.e.*, the square. Each and every point in either domains is completely defined and hence the error contribution is largely negligible. This suggests that the error margin was evolved mainly on the basis of material property scaling which has several sub – contributions as discussed below.

Nylon is a polymeric material and is used in the examples presented as its behavior is known to resemble the response of aluminum closely at room temperature regimes. Further, being a polymeric material, it is easy to manufacture in the standard SLS process thereby providing for a quick specimen fabrication. However, the fact remains that a polymer is used to evaluate a metal at low temperatures which is quite justified as none of the estimated parameters have temperature dependent properties. Hence, the chief contributions to the errors come from the assumption of linearity of the scales Λ_1 and Λ_2 which is reasonable but has errors inherent due to relative changes in magnitudes of the shapes and sizes of the structures involved. While the former scale is between smaller sizes of the product specimen and the model specimen, the latter is between the product and the model. Even though the functional relationship is the same as per the hypothesis, the errors creep in due to geometric effects such as distorted strain and stress concentrations.

In the second case, the Taylor series used to capture the material transformation has an inherent error term associated with it as the process produces close approximations in only small intervals. The reader should remember that the polynomial equation with a $R^2 = 0.9999$ is a curve obtained for the correction between the stress values of the actual and the predicted stress values of the model alone and does not incorporate any error that the Taylor series produces for the material map. While the initial correction negated any influence of size (discussed in the initial part of this discussion), the distortion produced due to the material approximation needs to be further evaluated. A similar correction needs to be done so that errors are brought down even further.

9. CONCLUSIONS: The technique of conformal mapping applied to ESM has been shown with very rigorous analytical definitions. The development and the applicability of the process have been presented. Some of the main conditions that influence the choice of ESM have been introduced along with the justification in migrating towards using complex variables and analytic functions as state variables. While the technique itself shows great promise, a more robust development is required to cut down on the error margin to a more acceptable magnitude. Largely though, the main hypothesis of using conformal mapping to compute scaling laws has been accomplished. Further, the power series expansion has generated scope for functional optimization to estimate optimal values for both the coefficients and the index of expansion.. Eventually, we would like to broaden the method to enough detail and generality to understand and map highly non – linear similarity problems within engineering efficacy.

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